LAGRANGIAN ZIGZAG COBORDISMS

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Abstract. We investigate an equivalence relation on Legendrian knots in the standard contact three-space defined by the existence of an interpolating zigzag of Lagrangian cobordisms. We compare this relation, restricted to genus-0 surfaces, to smooth concordance and Lagrangian concordance. We then study the metric monoid formed by the set of Lagrangian zigzag concordance classes, which parallels the metric group formed by the set of smooth concordance classes, proving structural results on torsion and satellite operators. Finally, we discuss the relation of Lagrangian zigzag cobordism to non-classical invariants of Legendrian knots.

1. Introduction

1.1. **Context and aims.** An essential framework in the study of contact and symplectic topology is to explore the boundary between flexibility and rigidity. A prominent locus for this study involves questions about the existence, (non-)uniqueness, and structure of exact, orientable Lagrangian concordances between Legendrian knots. Evidence for rigidity arises from the development of obstructions beyond the topological to the existence of such cobordisms [BLW22, BS18, EHK16, GJ19, Pan17, ST13] and the discovery of many symplectically distinct but smoothly isotopic Lagrangian concordances [CG22, CN22, Kál05]. Of particular importance for this paper is the structural fact that the relation on the set of Legendrian knots defined by Lagrangian concordance is reflexive and transitive, but not symmetric [Cha10, Cha15], though whether Lagrangian concordance yields a partial order is still an open question. This suggests that Lagrangian concordance is more akin to smooth ribbon concordance, which was recently shown by Agol to induce a partial order [Ago22], than to ordinary smooth concordance, which yields an equivalence relation.

The goal of this paper is to introduce a new relation we term **Lagrangian zigzag concordance** which lies between smooth and Lagrangian concordance, and to explore its structure and whether it still captures symplectic rigidity. Following [SVW21], we say that Legendrians Λ and Λ' are **Lagrangian**

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zigzag concordant if they are related by a sequence of connected Lagrangian concordances as in the following zigzag diagram:

$$\Lambda = \Lambda_0 \xrightarrow{L_1^<} \Lambda_{+,1} \xleftarrow{L_1^>} \Lambda_1 \quad \cdots \quad \Lambda_{n-1} \xrightarrow{L_n^<} \Lambda_{+,n} \xleftarrow{L_n^>} \Lambda_n = \Lambda' \tag{1.1}$$

See Section 2 for a precise definition. We note that Sarkar has defined a similar notion for ribbon concordance [Sar20].

The notions of zigzag cobordism and concordance allow us to pursue a closer analogy between contact and smooth knot theory. To begin, the Lagrangian zigzag concordance relation is an equivalence relation that refines the smooth concordance relation. Similarly to smooth knots, any two Legendrians with the same rotation number are related by zigzag cobordism [SVW21]. Thus, we may define a metric on the set $\mathcal{LZC}$ of zigzag concordance classes of Legendrians given by the minimum total genus g_{L} over all Lagrangian zigzag cobordisms (setting the distance between two Legendrians to be ∞ if they have different rotation numbers), which parallels the smooth concordance genus g_4 on the smooth concordance group $\mathcal{C}$.

The motivating questions for this paper may now come into sharper focus: First, how does the zigzag concordance relation refine the smooth concordance relation, and how do their structures compare? We can ask about both the metric structure and the algebraic structure of the relation. For example, what is the diameter of a component of $\mathcal{LZC}$ that corresponds to a fixed rotation number? Can we find interesting quasi-isometries of $\mathcal{LZC}$? Does connected sum induce a group structure on the set of zigzag concordance classes? Second, does the additional flexibility of the Lagrangian zigzag concordance relation still allow for symplectic rigidity? In particular, how does the zigzag concordance relation interact with existing non-classical invariants? We enumerate our efforts to answer these questions in the results detailed below.

While many (but not necessarily all) statements in the article hold for general (M, α) with essentially the same proof, for ease of exposition, we focus on $(\mathbb{R}^3, \mathrm{d}z - y\,\mathrm{d}x)$ throughout.

1.2. **Comparisons.** We first study the differences between smooth concordance, Lagrangian zigzag concordance, and Lagrangian concordance. It has already been shown that smooth and Lagrangian zigzag concordance do not coincide [SVW21, Example 6.6]. Just how similar these notions are, however, can be seen if we drop the assumption that the individual cobordisms in (1.1) are connected (even if the total cobordism is still required to be a cylinder). We call such an object a **disconnected Lagrangian zigzag-concordance**. It turns out that this notion coincides with smooth concordance:

Theorem 1.2. *The Legendrian knots Λ and Λ' are smoothly concordant if and only if they are disconnected Lagrangian zigzag concordant.*

On the other side, we show that Lagrangian zigzag concordance is coarser than Lagrangian concordance:

Theorem 1.3. *There exist a pair of Legendrian knots Λ and Λ' that are Lagrangian zigzag concordant but not Lagrangian concordant.*

1.3. **Structural results.** We next consider the metric properties of $\mathcal{LZC}$. Thinking of $\mathcal{LZC}$ as a graph weighted by g_{L}, we begin by investigating its basic properties:

Proposition 1.4. *The graph $\mathcal{LZC}$ has exactly one connected component for each rotation number. Each component has infinite diameter. The* 1*-link of each vertex of $\mathcal{LZC}$ is countably infinite.*

We then study $\mathcal{LZC}$ as a monoid under connected sum, both proving results and raising questions. For example, in contrast to the smooth case, it is unknown whether any non-trivial Lagrangian zigzag concordance class has an inverse. We open the topic by investigating the existence of torsion, proving a result that contrasts the case of $\mathcal{C}$:

Theorem 1.5. *No non-trivial amphicheiral Legendrian knot is* 2*-torsion in $\mathcal{LZC}$.*

Paralleling work of Cochran and Harvey [CH18, Proposition 4.1], we show that $\mathcal{LZC}$ is not δ-hyperbolic. Further, inspired by [CH18, Theorem 6.5], which shows that every winding-number-±1 satellite operator is a quasi-isometry of $\mathcal{C}$ to itself, we prove that $\mathcal{LZC}$ has a similar property:

Theorem 1.6. *Any winding-number-±1 satellite operator with rotation number* 0 *relative to the identity or reverse operator is a quasi-isometry of $\mathcal{LZC}$ to itself.*

The proof of this statement requires extending [SVW21, Proposition 3.1], which uses convex surface theory to obtain a span (i.e. a pair from a common lower bound) of decomposable Lagrangian cobordisms between any two null-homologous Legendrian knots in any contact 3-manifold. We establish the analogous statement (Lemma 3.13) for homologous knots in a possibly non-trivial homology class.

1.4. **Relation to non-classical invariants.** To better understand how non-classical invariants of Legendrian knots interact with zigzag concordance, we specialize to the Legendrian contact homology differental graded algebra (LCH DGA). The effective use of LCH often requires Lagrangian cobordisms to have Maslov number 0. Therefore, we prove a different strengthening of [SVW21] to produce Maslov-0 cospans for Legendrian links in $\mathbb{R}^3$:

Theorem 1.7 (cf. [SVW21, Theorem 1.1])**.** *If Λ and Λ' are oriented Legendrian links in the standard contact $\mathbb{R}^3$, all of whose components have vanishing rotation number, then there exist a Legendrian link Λ_+ and Maslov-0 Lagrangian cobordisms from Λ to Λ_+ and from Λ' to Λ_+.*

As we shall see in Section 5.2, however, the structure of the known non-classical invariants makes it difficult to use them to obstruct zigzag concordance. On one hand, this serves as motivation to develop a more refined understanding of these invariants under Lagrangian cobordism or to develop new styles of non-classical invariants. On the other, the precise nature of the failure of linearized LCH to provide an effective obstruction to zigzag concordance yields an interesting result, as follows. The set of linearized LCH, taken over all possible augmentations of the LCH DGA, is usually encoded by the set $\mathcal{P}_\Lambda$ of corresponding Poincaré–Chekanov polynomials. Zigzag cobordism yields a solution to the setwise geography problem for Poincaré–Chekanov polynomials, culminating in the following theorem:

Theorem 1.8. *For any finite set $\{p_1(t), \ldots, p_n(t)\}$ of polynomials with non-negative integral coefficients, there exist a Legendrian knot Λ and non-negative even integers $c_1, \ldots, c_n$ so that*

$$\{p_1(t) + p_1(t^{-1}) + t + c_1, \ldots, p_n(t) + p_n(t^{-1}) + t + c_n\} \subset \mathcal{P}_\Lambda.$$

This theorem generalizes a theorem of Melvin and Shrestha [MS05] that any single polynomial of the form $p(t) = q(t) + q(t^{-1}) + t$ can be realized as the linearized Legendrian contact homology of a knot with respect to an augmentation; see [BG22, BST15] for related results.

1.5. **Organization.** Throughout the paper, we assume familiarity with fundamental notions of Legendrian knots and contact topology as in [Etn05, Gei08, Tra21].

The article is organized as follows: We begin by defining Lagrangian zigzag cobordism in Section 2, and then comparing and contrasting that definition with smooth and Lagrangian cobordism. We then proceed to investigate the structure of the Lagrangian zigzag cobordism graph in Section 3, including proofs of Proposition 1.4, Theorem 1.5, and Theorem 1.6. Finally, we discuss interactions between zigzag cobordisms and non-classical invariants, first investigating the existence of Maslov-0 zigzag cobordisms in Section 4, then treating Legendrian contact homology in Section 5.

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2. The Lagrangian Zigzag Cobordism Relation

In this section, we formulate precise definitions of the Lagrangian zigzag cobordism and concordance relations on Legendrian links. We then compare the zigzag concordance relation to smooth concordance (proving Theorem 1.2) and Lagrangian concordance (proving Theorem 1.3).

2.1. **Definition of Lagrangian zigzag cobordism.** In order to define Lagrangian zigzag cobordism, we first recall the definition of a Lagrangian cobordism.

Definition 2.1. Let Λ_- and Λ_+ be Legendrian links in a contact 3-manifold $(M, \ker\alpha)$. An *exact Lagrangian cobordism* is an exact embedded orientable Lagrangian surface $L \subset (\mathbb{R} \times M, d(e^t\alpha))$ such that there exists T, satisfying that:

- $L \cap ((-\infty, -T) \times M) = (-\infty, -T) \times \Lambda_-$, denoted $\mathcal{E}_-(L)$;
- $L \cap ((T, \infty) \times M) = (T, \infty) \times \Lambda_+$, denoted $\mathcal{E}_+(L)$;
- There is a smooth function $f\colon L \to \mathbb{R}$ so that $df = e^t\alpha|_L$ and f is constant at each cylindrical end $\mathcal{E}_\pm(L)$; and
- $L \backslash (\mathcal{E}_-(L) \cup \mathcal{E}_+(L))$ is compact with boundary $(\{-T\} \times \Lambda_-) \cup (\{T\} \times \Lambda_+)$.

In this case we say that $\Lambda_- \prec \Lambda_+$ via the Lagrangian L or that L is a Lagrangian cobordism from Λ_- to Λ_+. A Lagrangian cobordism L between Legendrian knots is a **Lagrangian concordance** if it has genus 0.

Henceforth, unless specified, we will focus on the case $M = \mathbb{R}^3$ and $\alpha = \mathrm{d}z - y\mathrm{d}x$.

Classical invariants of Legendrian knots act as obstructions to Lagrangian cobordism. In particular, Chantraine [Cha10] proved:

Proposition 2.2 ([Cha10, Theorem 1.2])**.** *If* $\Lambda_- \prec \Lambda_+$ *via the Lagrangian* L*, then* $\mathrm{r}(\Lambda_-) = \mathrm{r}(\Lambda_+)$ *and* $\mathrm{tb}(\Lambda_+) - \mathrm{tb}(\Lambda_-) = -\chi(L)$.[1]

Lagrangian cobordisms may be constructed using traces of Legendrian isotopies or the attachment of Lagrangian 0- or 1-handles as in Figure 1 [BST15, EHK16]. A Lagrangian cobordism constructed using these three operations is called **decomposable**.

[1]The statement in fact holds for null-homologous knots $\Lambda_\pm$ in a general (M, α), with the rotation numbers taken with corresponding Seifert surfaces.

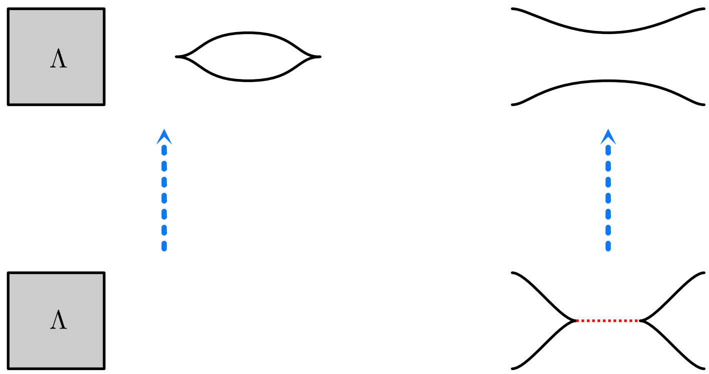

FIGURE 1. Attaching a Lagrangian 0-handle (left) and a 1-handle (right) to a Legendrian link.

To begin the precise definition of Lagrangian zigzag cobordism, let Λ and Λ' be two Legendrian links. A **Lagrangian cospan** $\mathbf{L}(\Lambda, \Lambda')$ consists of another Legendrian link Λ_+ and connected Lagrangian cobordisms L from Λ to Λ_+ and L' from Λ' to Λ_+, which we depict using the following diagram:

$$\begin{array}{ccccc} & & \Lambda_+ & & \\ & {\scriptstyle L}\nearrow & & \nwarrow{\scriptstyle L'} & \\ \Lambda & & & & \Lambda' \end{array}$$

Definition 2.3. Two Legendrian links Λ and Λ' are **Lagrangian zigzag cobordant**, denoted $\Lambda \sim \Lambda'$, if there exist Legendrian links

$$\Lambda = \Lambda_0, \Lambda_1, \ldots, \Lambda_{n-1}, \Lambda_n = \Lambda'$$

and Lagrangian cospans $\mathbf{L}(\Lambda_{i-1}, \Lambda_i)$ for $i = 1, \ldots, n$. The data defining a zigzag cobordism is denoted $\mathbb{L}(\Lambda, \Lambda')$.

If all of the Lagrangian cobordisms in $\mathbb{L}(\Lambda, \Lambda')$ are concordances, then Λ and Λ' are **Lagrangian zigzag concordant**, denoted $\Lambda \approx \Lambda'$.

See the diagram in (1.1) for a depiction of the data comprised by a Lagrangian zigzag cobordism. We note that both $\sim$ and $\approx$ are equivalence relations on the set of oriented Legendrian links.

Example 2.4. A Lagrangian cobordism L from Λ to Λ' induces a Lagrangian zigzag cobordism using the cylindrical cobordism from Λ' to itself:

$$\begin{array}{ccccc} & & \Lambda' & & \\ & {\scriptstyle L}\nearrow & & \nwarrow{\scriptstyle =} & \\ \Lambda & & & & \Lambda' \end{array}$$

Example 2.5. Suppose that Λ and Λ' are decomposably Lagrangian slice, i.e. there are decomposable Lagrangian concordances L and L' from the maximal unknot Υ to Λ and Λ', respectively. For example, the Legendrian realizations of the $m(9_{46})$ and $m(12n_{768})$ knots in Figure 2 are decomposably slice [CNS16].

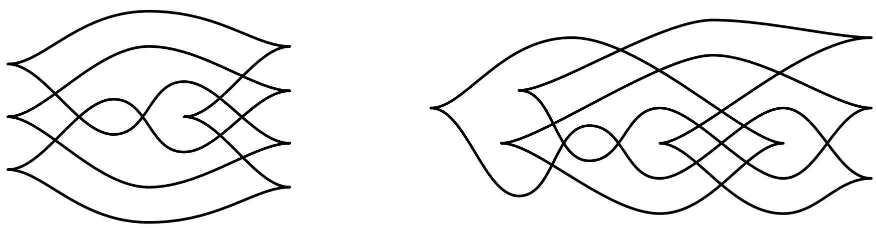

FIGURE 2. Legendrian realizations of the Lagrangian slice knots $m(9_{46})$ (left) and $m(12n_{768})$ (right).

The Legendrians Λ and Λ' are Lagrangian zigzag concordant:

$$\Lambda \xrightarrow{=} \Lambda \xleftarrow{L} \Upsilon \xrightarrow{L'} \Lambda' \xleftarrow{=} \Lambda'$$

We attach several quantities to a Lagrangian zigzag cobordism. The **Euler characteristic** $\chi(\mathbb{L}(\Lambda, \Lambda'))$ of a Lagrangian zigzag cobordism is simply the sum of the Euler characteristics of the constituent cobordisms; the genus of a Lagrangian zigzag cobordism is then defined as usual. We use these notions to define a relative genus between two Legendrians:

Definition 2.6. Given two Legendrian knots Λ and Λ', their **relative Lagrangian zigzag cobordism genus** $g_{\mathrm{L}}(\Lambda, \Lambda')$ is the minimal genus of all Lagrangian zigzag cobordisms between Λ and Λ'.

Remark 2.7. It is clear that the relative Lagrangian zigzag cobordism genus descends to zigzag concordance classes.

Remark 2.8. In [SVW21, Section 6], it was shown that the smooth relative 4-genus is a lower bound on the relative Lagrangian zigzag cobordism genus, and that the bound is not always realized. In particular, the relative Lagrangian zigzag cobordism genus between the maximal unknot and its double stabilization is 1 while the relative smooth genus is clearly 0.

The classical invariants interact nicely with Lagrangian zigzag cobordism and its genus.

Proposition 2.9. *Let* Λ *and* Λ' *be Legendrian knots. Suppose that* $\Lambda \sim \Lambda'$*; then* $\mathrm{r}(\Lambda) = \mathrm{r}(\Lambda')$ *and*

$$\left|\mathrm{tb}(\Lambda) - \mathrm{tb}(\Lambda')\right| \leq 2g_{\mathrm{L}}(\Lambda, \Lambda').$$

In particular, if $\Lambda \approx \Lambda'$*, then* $\mathrm{tb}(\Lambda) = \mathrm{tb}(\Lambda')$*.*

Proof. The invariance of the rotation number follows from Proposition 2.2, which implies that all Legendrians in $\mathbb{L}(\Lambda, \Lambda')$ have the same rotation number. The bound on the difference between Thurston–Bennequin numbers follows from the estimate

$$\begin{aligned}
\left|\mathrm{tb}(\Lambda) - \mathrm{tb}(\Lambda')\right| &= \left|\sum_{i=1}^{n} \left(\chi(L_i^{<}) - \chi(L_i^{>})\right)\right| \\
&\leq \sum_{i=1}^{n} \left(-\chi(L_i^{<}) - \chi(L_i^{>})\right) \\
&= -\chi(\mathbb{L}(\Lambda, \Lambda') \\
&= 2g(\mathbb{L}(\Lambda, \Lambda')).
\end{aligned}$$

We used the fact that each constituent cobordism has at least two boundary components, and hence has non-positive Euler characteristic; we also used the fact that the Legendrians at the ends are connected in the last line. □

Finally, we define the **Maslov number** $\mu(\mathbb{L}(\Lambda, \Lambda'))$ of a Lagrangian zigzag cobordism to be the greatest common divisor of the Maslov numbers of the constituent Lagrangian cobordisms. Of particular interest are zigzag cobordisms with Maslov number 0. See Section 4 for further discussion of Maslov-0 zigzag cobordisms.

2.2. **Comparison to smooth cobordism.** With the definition and basic properties of Lagrangian zigzag concordance in hand, we next explore its relationship with smooth concordance. In the proof of Proposition 2.9, we have seen that the connectedness assumption leads to the Thurston-Bennequin number being invariant under Lagrangian zigzag concordance, indicating some level of rigidity in the relation. In this section, we will prove Theorem 1.2, which shows that without the connectedness condition, the notion of Lagrangian zigzag concordance is quite flexible, reducing to its smooth counterpart.

Write $\Lambda \approx_{\mathrm{d}} \Lambda'$ if Λ and Λ' satisfy the definition of Lagrangian zigzag concordance, but with the intermediate Lagrangians allowed to be disconnected even though the total underlying smooth cobordism is still a cylinder; in this case, we say that Λ and Λ' are **disconnected Lagrangian zigzag concordant**.

The flexibility in disconnected Lagrangian zigzag concordance arises from the following construction.

Lemma 2.10. *A Legendrian knot* Λ *is disconnected Lagrangian zigzag concordant to its double stabilization* $S_{+-}(\Lambda)$.

Proof. The zigzag concordance is constructed in Figure 3. □

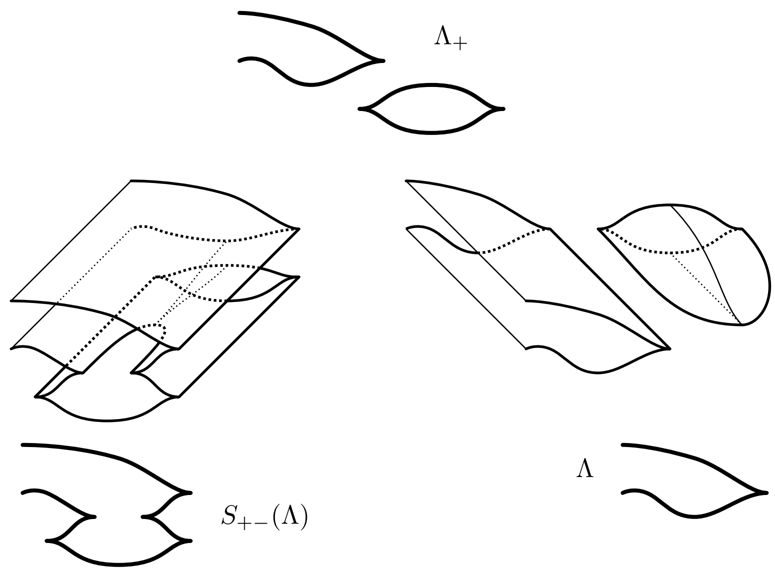

FIGURE 3. A disconnected Lagrangian zigzag concordance between Λ and $S_{+-}(\Lambda)$. The Lagrangian on the left is constructed from a 1-handle, and the concordance on the right comes from a (cancelling) 0-handle. The surfaces pictured are front diagrams of the Legendrian lifts of the exact Lagrangian cobordisms.

Corollary 2.11. *If two Legendrian links* Λ_1 *and* Λ_2 *are smoothly isotopic and the corresponding components have the same rotation numbers, then they are disconnected Lagrangian zigzag concordant.*

Proof. Under the hypotheses of the corollary, Fuchs and Tabachnikov [FT97, Theorem 4.4] show that there exist $m, n \in \mathbb{Z}_{\geq 0}$ such that the m-fold double stabilization $S^m_{+-}(\Lambda_1)$ is Legendrian isotopic to $S^n_{+-}(\Lambda_2)$. Lemma 2.10 shows that $\Lambda_1 \approx_{\mathrm{d}} S^m_{+-}(\Lambda_1)$ and that $\Lambda_2 \approx_{\mathrm{d}} S^n_{+-}(\Lambda_2)$, while the fact that Legendrian isotopy induces a Lagrangian cobordism shows that $S^m_{+-}(\Lambda_1) \approx_{\mathrm{d}} S^n_{+-}(\Lambda_2)$. The proof then follows from the transitivity of the Lagrangian zigzag concordance relation. □

Proof of Theorem 1.2. The reverse direction is clear, so we need only prove that if $F \subset \mathbb{R}^3 \times [0,1]$ is a smooth concordance between Λ and Λ', then Λ and

Λ' are disconnected Lagrangian zigzag concordant. Let r denote the common rotation number of Λ and Λ'.

The first step is to decompose F into a sequence of elementary cobordisms

$$F = F_1 \odot \cdots \odot F_n,$$

where each F_i is the trace of an isotopy or the attachment of an m-handle for $m = 0, 1, 2$. Denote by K_{i-1} and K_i the links that form the bottom and top boundaries, respectively, of F_i.

Let γ be an oriented smooth embedded path in F that begins on $\Lambda = K_0$, ends on $\Lambda' = K_n$, intersects each homologically non-trivial component of the simple multicurve $K_i \subset F$ exactly once, and does not intersect any null-homologous component. Such a path exists by an argument that successively removes innermost arcs that arise when a candidate path intersects a component of K_i at least twice. A consequence of this construction is that if F_i is the attachment of a 0-handle (resp. a 2-handle) and K_i^0 is the component of K_i created by the 0-handle (resp. K_{i-1}^2 is the component erased by the 2-handle), then γ does not intersect K_i^0 (resp. K_{i-1}^2). We say that a Legendrian representative Λ_i of K_i is **γ-compatible** if the rotation number is r for all components that γ intersects in an upward direction with respect to the $[0, 1]$-component, is $-r$ for all components γ intersects in a downward direction, and is 0 otherwise.

The proof now proceeds inductively, one elementary cobordism at a time, by showing that if Λ_{i-1} is a γ-compatible Legendrian representative of the link K_{i-1}, then there exists a γ-compatible Legendrian representative Λ_i of K_i with $\Lambda_{i-1} \approx_{\mathrm{d}} \Lambda_i$. We may start the inductive process since $K_0 = \Lambda$ is a γ-compatible Legendrian knot by construction.

We prove the inductive claim in four cases:

Isotopy: Choose a Legendrian representative Λ_i of K_i. By further stabilizing each component of Λ_i appropriately, we may assume that Λ_i is γ-compatible. In particular, we see that corresponding components of Λ_{i-1} and Λ_i have the same rotation numbers. Corollary 2.11 then implies that Λ_{i-1} and Λ_i are disconnected Lagrangian zigzag concordant.

0-handle: Simply take Λ_i to be the result of attaching a Lagrangian 0-handle to Λ_{i-1}. Since the component of K_i birthed by a 0-handle cannot intersect γ, we conclude that Λ_i is γ-compatible if Λ_{i-1} is.

1-handle: Let $\beta : [-1, 1] \times [-\delta, \delta] \to \mathbb{R}^3$ be the embedded band along which the 1-handle is attached, with $\beta(\{-1, 1\} \times [-\delta, \delta]) \subset \Lambda_{i-1}$. Perform an isotopy on β supported in $(-2\epsilon, 2\epsilon) \times [-\delta, \delta]$, for sufficiently small $\epsilon > 0$, so that along $[-\epsilon, \epsilon] \times [\delta, \delta]$, the result of the isotopy matches the band in the standard Lagrangian 1-handle (see [Dim16, EHK16]. Call the resulting embedding of the band β'.

Let Λ'_{i-1} be a Legendrian link that agrees with Λ_{i-1} off of $\beta(\{-1,1\}\times[-\delta,\delta])$, contains $\beta'(\{-\epsilon,\epsilon\}\times[-\delta,\delta])$, and C^0 approximates the curves defined by β' that join the endpoints of the curves above. By construction, Λ_{i-1} and Λ'_{i-1} are smoothly isotopic. Add small stabilizations to Λ'_{i-1} if necessary to ensure that rotation numbers of corresponding components of Λ_{i-1} and Λ'_{i-1} agree. By the first case, above, we know that Λ_{i-1} and Λ'_{i-1} are disconnected Lagrangian zigzag concordant.

We then let Λ_i be the result of attaching the Lagrangian 1-handle defined by β' to Λ'_{i-1}. To verify that Λ_i is the desired Legendrian link, we first note that it is, indeed, a representative of the smooth knot K_i since it was obtained from K_{i-1} by attaching a 1-handle along a band isotopic to the original band. Second, we check that Λ_i is γ-compatible. The key property here is that the sum of the rotation numbers of the components on one end of the Lagrangian 1-handle cobordism is equal to the rotation number of the component on the other end. We conclude that Λ_i is γ-compatible, as either γ intersects two components on one end of the cobordism in opposite directions (and hence the rotation number on the other end, which cannot intersect γ, is $r-r=0$) or intersects one component on either end of the cobordism (and hence the rotation number of the non-intersected component must be 0 while the rotation numbers of the other components are both $\pm r$).

2-handle: We first note that the component Λ^2_{i-1} of Λ_{i-1} that is erased by the 2-handle does not intersect γ, and hence has rotation number 0. Further, the component Λ^2_{i-1} is unknotted and unlinked from the rest of Λ_{i-1}. Thus, by Corollary 2.11, the link Λ_{i-1} is Lagrangian zigzag concordant to a Legendrian link Λ'_{i-1} with the component Λ^2_{i-1} replaced by a maximal unknot. Further, Λ_i is just the link Λ_{i-1} with Λ^2_{i-1} removed. Thus, by attaching a Lagrangian 0-handle to Λ_i, we obtain a Lagrangian cobordism from Λ_i to Λ'_{i-1}. It follows that we have $\Lambda_{i-1}\approx_{\mathrm{d}}\Lambda'_{i-1}\approx\Lambda_i$. The γ-compatibility of Λ_i follows immediately.

This completes the inductive argument. Together with transitivity of the relation $\approx_{\mathrm{d}}$, this suffices to prove the proposition since Λ_n is a Legendrian representative of the knot $K_n=\Lambda'$ and the γ-compatibility of Λ_n implies that Λ_n and Λ' have the same rotation number, so Corollary 2.11 shows that $\Lambda_n\approx_{\mathrm{d}}\Lambda'$. □

2.3. **Comparison to Lagrangian cobordism.** Remark 2.8, Proposition 2.9, and Theorem 1.2 show that the Lagrangian zigzag concordance relation is more rigid than smooth concordance. On the other hand, Theorem 1.3, which we will prove in this section, shows that Lagrangian zigzag concordance is more flexible that Lagrangian concordance. In particular, we find a pair of Legendrian knots Λ and Λ' that are Lagrangian zigzag concordant but not

Lagrangian concordant in either direction. Our pair of knots is the one in Example 2.5 and Figure 2, which we already know to be Lagrangian zigzag concordant.

To show these knots are not Lagrangian concordant in either direction, the first ingredient is Pan's obstruction to Lagrangian cobordisms via ruling polynomials [Pan17], which are a type of skein relation invariant of Legendrian knots. See the survey article [Sab21] for an introduction to the notion of a ruling of a Legendrian link.

Theorem 2.12 ([Pan17, Corollary 1.7])**.** *Suppose that there exists a spin exact Maslov-0 Lagrangian concordance from Λ_- to Λ_+. The ruling polynomials satisfy the following inequality for any prime power q:*

$$R_{\Lambda_-}(q^{1/2} - q^{-1/2}) \leq R_{\Lambda_+}(q^{1/2} - q^{-1/2}).$$

Note that we may apply the theorem to obstruct concordances between any two Legendrians with rotation number 0, as a Lagrangian concordance between two knots of rotation number 0 automatically has Maslov number 0; further, any orientable 2-manifold has a spin structure that may be restricted to spin structures along the boundary.

The second ingredient is the Legendrian satellite $\Sigma(\Lambda, \Pi)$ of a pattern $\Pi \subset J^1S^1$ around a companion $\Lambda \subset \mathbb{R}^3$ (see [EV18, Ng01, NT04]) and Cornwell, Ng, and Sivek's construction of satellites of concordances [CNS16].

Theorem 2.13 ([CNS16, Theorem 2.4])**.** *Let Π be a Legendrian pattern in J^1S^1. If $\Lambda_- \prec \Lambda_+$, then $\Sigma(\Lambda_-, \Pi) \prec \Sigma(\Lambda_+, \Pi)$.*

There is a generalization of this theorem to Lagrangian cobordisms in [GSY22], though we will not need the full power of the results therein.

Proof of Theorem 1.3. As shown in Example 2.5, the Legendrian knots Λ and Λ' from Figure 2 are Lagrangian zigzag concordant.

On the other hand, it suffices to show that the tb-twisted Whitehead doubles $\mathrm{Wh}(\Lambda)$ and $\mathrm{Wh}(\Lambda')$ are not Lagrangian concordant in either direction using Theorem 2.12; the contrapositive of Theorem 2.13 then implies that Λ and Λ' are also not concordant in either direction.

A tedious but straightforward computation shows that

$$R_{\mathrm{Wh}(\Lambda)}(z) = 5 + 3z^2,$$
$$R_{\mathrm{Wh}(\Lambda')}(z) = 2 + 3z^2 + 3z^4 + z^6.$$

We then compute that

$$R_{\mathrm{Wh}(\Lambda)}\left(\sqrt{2} - 1/\sqrt{2}\right) > R_{\mathrm{Wh}(\Lambda')}\left(\sqrt{2} - 1/\sqrt{2}\right),$$
$$R_{\mathrm{Wh}(\Lambda)}\left(\sqrt{3} - 1/\sqrt{3}\right) < R_{\mathrm{Wh}(\Lambda')}\left(\sqrt{3} - 1/\sqrt{3}\right).$$

The result follows. □

3. Global structure of the zigzag cobordism relation

In this section, we study the global structure, metric, and monoidal properties of the Lagrangian zigzag cobordism relation. After constructing a weighted graph with edge metric that describes the zigzag cobordism relation, we explore

- The structure of the weighted graph, including its connectivity, diameter, and links of vertices (Section 3.1);
- The monoidal structure of the zigzag cobordism relation (Section 3.2); and
- Quasi-isometries of the graph given by the satellite construction (Section 3.3).

3.1. **The Lagrangian zigzag cobordism graph.** To frame the study of the global structure of the Lagrangian zigzag cobordism relation, we take inspiration from Cochran and Harvey's study [CH18] of the smooth knot concordance group as a metric group with distance defined by the minimal smooth relative 4-genus g_4 between concordance classes of knots. In particular, we make the following definition:

Definition 3.1. The Lagrangian zigzag cobordism graph $\mathcal{LZC}$ is the weighted graph defined by the following:

- Its vertices correspond to equivalence classes of oriented Legendrian knots under the Lagrangian zigzag concordance relation $\approx$.
- A single edge exists between equivalence classes $[\Lambda_1]$ and $[\Lambda_2]$ if $\Lambda_1 \sim \Lambda_2$.
- The weight of an edge between $[\Lambda_1]$ and $[\Lambda_2]$ is $g_{\mathrm{L}}(\Lambda_1, \Lambda_2)$.

Since g_{L} is clearly subadditive and symmetric, the weighted graph $\mathcal{LZC}$ is also a metric space.

We will henceforth drop the brackets from the notation for equivalence classes of Lagrangian zigzag concordance, denoting such an equivalence class by a representative.

The remainder of this subsection is devoted to proving fundamental properties of the graph $\mathcal{LZC}$, as encapsulated in Proposition 1.4. First, the connectivity of $\mathcal{LZC}$ is determined by the main result of [SVW21]:

Proposition 3.2. *The graph $\mathcal{LZC}$ has exactly one connected component for each rotation number.*

Next, we show that each component of $\mathcal{LZC}$ has infinite diameter. While this may be proven using the fact that the underlying smooth cobordism graph has infinite diameter with respect to g_4—for example, one may take connected sums with $(2, n)$-torus knots—the proposition below shows that $\mathcal{LZC}$ contains finer structure than the smooth concordance group as a metric space.

Proposition 3.3. *For any Legendrian Λ and $n \in \mathbb{N}$, there exists a Legendrian Λ_n such that $g_{\mathrm{L}}(\Lambda, \Lambda_n) = n$ and $g_4(\Lambda, \Lambda_n) = 0$.*

Proof. Let Λ' be the maximal Legendrian representative of the $m(6_1)$ knot, pictured in Figure 4.

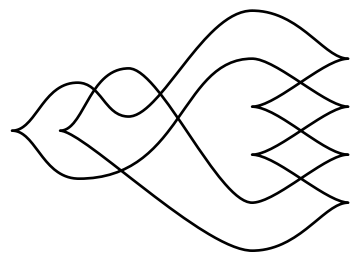

FIGURE 4. The maximal Legendrian $m(6_1)$ knot.

This knot is smoothly slice, but since its Thurston–Bennequin number is -3, it is not Lagrangian slice. Consider $\Lambda_n = \Lambda \sharp n\Lambda'$. On one hand, this knot satisfies $g_4(\Lambda, \Lambda_n) = 0$. On the other hand, we may compute that

$$|\mathrm{tb}(\Lambda) - \mathrm{tb}(\Lambda_n)| = 2n,$$

and hence Proposition 2.9 implies that $g_{\mathrm{L}}(\Lambda, \Lambda_n) \geq n$. □

Local properties of the $\mathcal{LZC}$ graph may be stated using the 1-link of a vertex Λ, which we define to be the set of vertices that have distance exactly 1 from Λ.

Proposition 3.4. *Given a Legendrian knot Λ, the cardinality of its 1-link is countably infinite. In particular, there exists a sequence of Legendrian knots Λ_i so that $g_{\mathrm{L}}(\Lambda, \Lambda_i) = 1$ but $\Lambda_i \not\approx \Lambda_j$ for $i \neq j$.*

Proof. Given $i > 0$, let Λ'_i be the Legendrian representative of the pretzel knot $K_i = P(-3, -5, -(2i+1))$. As illustrated in Figure 5, there is a genus 1 Lagrangian cobordism from the maximal Legendrian unknot Υ to Λ'_i. By [SVW21, Lemma 6.7], we then have $g_{\mathrm{L}}(\Upsilon, \Lambda'_i) = 1$.

It is straightforward to compute that the Alexander polynomial of K_i is given by

$$\Delta_{K_i}(t) = (4i+6)t^2 + (8i+11)t + 4i + 6,$$

which has degree 2 and is irreducible and distinct for each i. Then for any K_s, $r \neq s$,

$$\Delta_{K_r \sharp K_s}(t) = \Delta_{K_r}(t)\Delta_{K_s}(t) \neq t^k f(t) f(t^{-1})$$

for any polynomial f and integer k. Thus no two such pretzel knots are smoothly concordant.

Now, given a Legendrian Λ, let $\Lambda_i = \Lambda \sharp \Lambda'_i$. Then, again using [SVW21, Lemma 6.7], we have $g_{\mathrm{L}}(\Lambda, \Lambda_i) = 1$ but $\Lambda_i \not\approx \Lambda_j$ for $i \neq j$ since they cannot be smoothly concordant. □

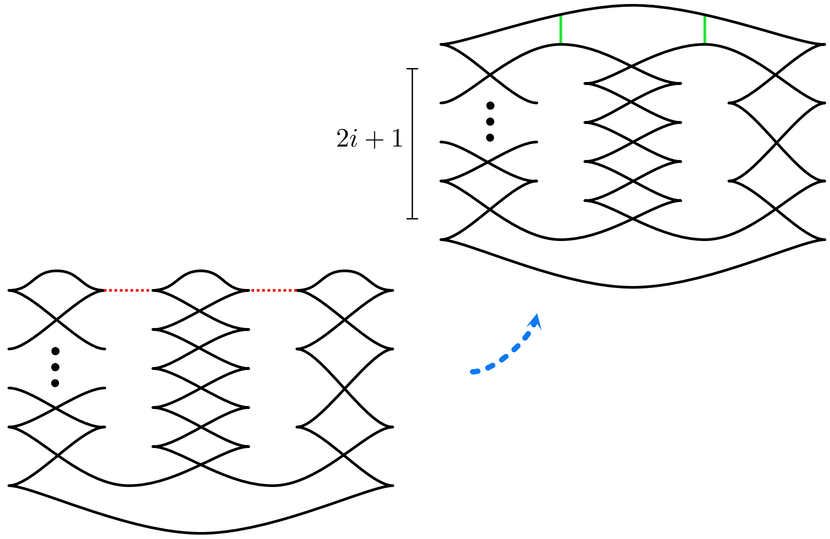

FIGURE 5. By attaching two 1-handles to a maximal Legendrian unknot, we obtain a genus 1 Lagrangian cobordism from that unknot to Λ_i'.

Proof of Proposition 1.4. The proposition is a combination of Proposition 3.2, Proposition 3.3, and Proposition 3.4. ☐

3.2. **Monoidal structure.** In this section, we turn to the algebraic structure of $\mathcal{LZC}$. The monoidal structure on $\mathcal{LZC}$ is defined by connected sum $\sharp$ [EH03]. That the connected sum operation is well defined on $\mathcal{LZC}$ follows from [GSY22, Corollary 5.1].

In the smooth setting, the connected sum becomes a group operation when we descend to concordance classes: In particular, inverses exist since the connected sum of a knot and its mirror reverse is smoothly slice. In the Legendrian setting, while we still have a unit given by the maximal unknot Υ, and the connected sum operation is still associative, evidence points to the idea that inverses *never* exist. Broadly speaking, the obstruction is that Legendrian realizations of smooth mirrors are not well behaved. We discuss two indications that this is, indeed, the case.

To set notation, let Λ be a Legendrian knot and Λ' be a Legendrian realization of its mirror reverse; of course, $\Lambda \sharp \Lambda'$ is smoothly slice. First, if Λ is stabilized, then so is $\Lambda \sharp \Lambda'$, and hence it cannot be Lagrangian concordant to Υ. It is unclear, however, if $\Lambda \sharp \Lambda'$ is Lagrangian zigzag concordant to Υ, especially as it is unknown whether stabilizations are preserved under Lagrangian zigzag concordance (though we know that they are *not* preserved under Lagrangian zigzag *cobordism*). Second, assuming that Λ and Λ' have maximal tb, a necessary condition for $\Lambda \sharp \Lambda'$ to be Lagrangian slice would be for $\operatorname{tb}(\Lambda) = -\operatorname{tb}(\Lambda')$, which would imply that $\operatorname{tb}(\Lambda \sharp \Lambda') = -1$. This condition, however, is not satisfied for any knot of 10 crossings or fewer [LM]. That said, it may be possible to find a Legendrian realization Λ'' of a smooth knot concordant to Λ' that has the correct tb.

Open Question 3.5. Does any non-trivial Lagrangian zigzag concordance class have an inverse with respect to the connected sum operation? If not, does the connected sum operation have the cancellation property?

As a first step in investigating the algebraic structure of the $\mathcal{LZC}$ monoid, we study torsion (or the lack thereof). The underlying result relates the maximal Thurston–Bennequin number $\overline{\operatorname{tb}}$ and the arc index α of a smooth knot and its mirror.

Proposition 3.6. *For any smooth knot K, we have*

$$\overline{\operatorname{tb}}(K) + \overline{\operatorname{tb}}(m(K)) = -\alpha(K).$$

Before beginning the proof, we need to set some notation. If $\mathbb{G}$ is a grid diagram for K, then we denote the $(\pi/2)$-rotation by $m(\mathbb{G})$, which is a grid diagram for $m(K)$. Further, the Legendrian front associated to $\mathbb{G}$ is denoted $\Lambda_{\mathbb{G}}$.

Proof of Proposition 3.6. Let $\mathbb{G}$ be a minimal grid diagram for K. By [DP13, Corollary 3], the associated front diagram $\Lambda_{\mathbb{G}}$ realizes $\overline{\operatorname{tb}}(K)$. The same is true for $\Lambda_{m(\mathbb{G})}$ and $\overline{\operatorname{tb}}(m(K))$.

When we compute $\operatorname{tb}(\Lambda_{\mathbb{G}}) + \operatorname{tb}(\Lambda_{m(\mathbb{G})})$, the contributions from the crossings cancel, as each positive (resp. negative) crossing in $\Lambda_{\mathbb{G}}$ is a negative (resp. positive) crossing in $\Lambda_{m(\mathbb{G})}$. Thus, $\operatorname{tb}(\Lambda_{\mathbb{G}}) + \operatorname{tb}(\Lambda_{m(\mathbb{G})})$ is equal to half the total number of cusps in $\Lambda_{\mathbb{G}}$ and $\Lambda_{m(\mathbb{G})}$. Each X or O in the grid diagram $\mathbb{G}$ yields a cusp in exactly one of $\Lambda_{\mathbb{G}}$ or $\Lambda_{m(\mathbb{G})}$, and hence we see that the total number of cusps in $\Lambda_{\mathbb{G}}$ and $\Lambda_{m(\mathbb{G})}$ is equal to twice $\alpha(K)$. □

Corollary 3.7. *No non-trivial amphicheiral Legendrian knot is 2-torsion in $\mathcal{LZC}$.*

Proof. Suppose that Λ is amphicheiral and 2-torsion. Since Λ is amphicheiral, Proposition 3.6 shows that $\alpha(K) \leq -2\operatorname{tb}(\Lambda)$. On the other hand, since Λ is 2-torsion, we have $-1 = \operatorname{tb}(\Lambda \sharp \Lambda) = 2\operatorname{tb}(\Lambda) + 1$, and hence that $2\operatorname{tb}(\Lambda) = -2$. Thus, we see that $\alpha(\Lambda) \leq 2$, and hence that Λ is an unknot. □

3.3. **Metric structure.** In this subsection, we state some properties of $\mathcal{LZC}$ as a metric space (or equivalently, a weighted graph). The main theorems here are inspired by their smooth analogues in [CH18], and we point the reader to

[CH18] for relevant definitions, e.g. of quasi-isometry and related terms. We begin by establishing the non-hyperbolicity of $\mathcal{LZC}$.

Theorem 3.8 (cf. [CH18, Theorem 4.1])**.** *Given $n \geq 1$, there exists a subspace of $\mathcal{LZC}$ that is quasi-isometric to $\mathbb{R}^n$. Consequently, $\mathcal{LZC}$ cannot be isometrically embedded in a finite product of δ-hyperbolic spaces.*

Proof. In the proof of [CH18, Theorem 4.1], knots $K_1, \dots, K_n$ that are linearly independent in the smooth concordance group $\mathcal{C}$ are chosen, together with Tristram signature functions $\sigma_j \colon \mathcal{C} \to \mathbb{Z}$, $1 \leq j \leq n$, that satisfy

$$g_4(K) \geq \frac{1}{2}\left|\sigma_j(K)\right| \text{ for all } K, \qquad \sigma_j(K_i) = 2\delta_{ij}. \tag{3.9}$$

The goal there is then to prove, for $\vec{x} = (x_1, \dots, x_n), \vec{y} = (y_1, \dots, y_n) \in \mathbb{Z}^n$, the inequalities

$$\frac{1}{n} d_{\mathrm{t}}(\vec{x}, \vec{y}) \leq g_4(\vec{x}, \vec{y}) \leq n d_{\mathrm{t}}(\vec{x}, \vec{y}), \tag{3.10}$$

where d_{t} is the taxicab metric on $\mathbb{Z}^n$, and $g_4(\vec{x}, \vec{y})$ is the slice genus distance between $x_1 K_1 \sharp \cdots \sharp x_n K_n$ and $y_1 K_n \sharp \cdots \sharp y_n K_n$. To complete that proof, the left inequality of (3.10) follows from (3.9), while the right inequality follows from the subadditivity and symmetry of metrics.

In the present context, we follow the same proof, limiting $\vec{x}$ and $\vec{y}$ to $\mathbb{Z}^n_{\geq 0}$ and choosing Legendrian representatives Λ_i of K_i. Then the Legendrians Λ_i are guaranteed to be linearly independent in $\mathcal{LZC}$, and the goal is to prove that

$$\frac{1}{n} d_{\mathrm{t}}(\vec{x}, \vec{y}) \leq g_{\mathrm{L}}(\vec{x}, \vec{y}) \leq n d_{\mathrm{t}}(\vec{x}, \vec{y}), \tag{3.11}$$

where $g_{\mathrm{L}}(\vec{x}, \vec{y})$ is the relative Lagrangian genus between $x_1 \Lambda_1 \sharp \cdots \sharp x_n \Lambda_n$ and $y_1 \Lambda_1 \sharp \cdots \sharp y_n \Lambda_n$. The left inequality of (3.11) follows from (3.9) as in the proof of [CH18, Theorem 4.1], with the additional observation that $g_{\mathrm{L}}(\Lambda) \geq g_4(\Lambda)$. ☐

Next, we prove Theorem 1.6, which states that winding-number-± 1 satellite operators with rotation number 0 relative to the identity or reverse operator are self-quasi-isometries of $\mathcal{LZC}$. To clarify this statement, every satellite operator corresponds to a pattern Legendrian link in $J^1 S^1$; for example, the identity operator corresponds to the core of $J^1 S^1$. A pair of homologous links $\Pi_1, \Pi_2 \in J^1 S^1$ cobound a Seifert surface S, meaning that $\partial S = \Pi_1 \sqcup -\Pi_2$, giving a relative rotation number $\mathrm{r}(\Pi_1 \sqcup -\Pi_2)$ when they are both Legendrian. *A priori,* this relative rotation number depends on the homology class of the Seifert surface; but since $H_2(J^1 S^1) = 0$, it is in fact well defined.

As in [CH18], we use the following lemma to transform the problem into proving that such operators are a bounded distance from known quasi-isometries.

Lemma 3.12 ([CH18, Lemma 6.2])**.** *Let (X, d) and (X', d') be two metric spaces. Suppose that $f \colon X \to X'$ is a quasi-isometric embedding. If $g \colon X \to X'$ is within a bounded distance from f, then g is a quasi-isometric embedding. Further, if f is a quasi-isometry, then so is g.*

To prove the bounded distance, we first extend [SVW21, Proposition 3.1] to links in J^1S^1 that are not necessarily null-homologous.

Lemma 3.13 ([SVW21, Proposition 3.1])**.** *Suppose that Λ_1 and Λ_2 are Legendrian links in (M, α) with $[\Lambda_1] = [\Lambda_2] \in H_1(M)$ and relative rotation number $r_{[\Sigma]}(\Lambda_1 \sqcup -\Lambda_2) = 0$ with respect to some Seifert surface Σ for $\Lambda_1 \sqcup -\Lambda_2$. Then there exists a Lagrangian zigzag cobordism between Λ_1 and Λ_2.*

Proof. The proof of Lemma 3.13 follows an argument that closely mirrors the proof of [SVW21, Lemma 3.5]. In what follows, we discuss the basic strategy but point the reader to [SVW21] for a more detailed treatment.

We can assume without loss of generality that both Λ_1 and Λ_2 are knots. If that were not the case, then one can perform Legendrian surgery along a collection of Legendrian arcs in their complement to separately join each of their components.

We now consider a Seifert surface Σ for the link $\Lambda_1 \sqcup -\Lambda_2$. After possibly double-stabilizing Λ_1 and Λ_2 to achieve negative twisting of ξ along each component of $\partial\Sigma$, we can isotope Σ relative to the boundary to be convex.

The idea now is to view Σ in disk–band form by choosing an arc-basis $\{a_1, \dots, a_g, a_{g+1}\}$ for Σ consisting of a collection of properly embedded arcs in Σ, such that Λ_2 intersects only a_{g+1} and does so in a single point. The proof of [SVW21, Lemma 3.5] details how, via Legendrian ambient surgery (and therefore Lagrangian cobordism), one can decompose Σ by cutting along the arcs $a_1, \dots, a_g$.

At the end of the decomposition process, we obtain a Legendrian knot Λ which is Lagrangian cobordant to Λ_1. Moreover, Λ and Λ_2 cobound a convex annulus A (the portion of Σ bounded by Λ and Λ_2). This implies that Λ and Λ_2 are smoothly isotopic. By construction, $r_{[A]}(\Lambda \sqcup -\Lambda_2) = 0$, implying that after sufficiently many double stabilizations of each, Λ and Λ_2 become Legendrian isotopic. In turn, Λ_2 is Lagrangian zigzag cobordant to Λ, which is itself Lagrangian zigzag cobordant to Λ_1, finishing the argument. ☐

We now state an immediate corollary that will be useful in the arguments that follow. Let $C_{n,1}$ be the Legendrian in J^1S^1 depicted in Figure 6. Denote by $C_{n,1}^m$ the Legendrian knot obtained by stabilizing $C_{n,1}$ either positively or negatively $|m|$ times, depending on the sign of m.

Corollary 3.14. *A Legendrian link Π in J^1S^1 is Lagrangian zigzag cobordant to $C_{n,1}^m$ if and only if n is the winding number of Π and m is the unique number for which $\mathrm{r}(\Pi \sqcup -C_{n,1}^m) = 0$.*

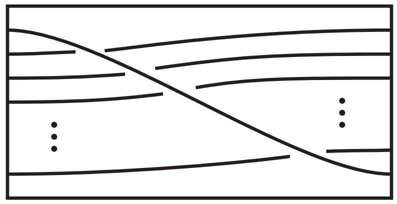

FIGURE 6. The Legendrian knot $C_{n,1} \in J^1S^1$.

Proof. Obviously, $[\Pi] = [C_{n,1}] \in H_1(J^1S^1)$. It is also clear that the number m as described as unique. By Lemma 3.13, the condition $\mathrm{r}(\Pi \sqcup -C^m_{n,1})$ implies that Π and $C^m_{n,1}$ are Lagrangian zigzag cobordant. □

Proposition 3.15 (cf. [CH18, Proposition 6.3])**.** *Suppose that* $\Pi\colon (\mathcal{LZC}, g_{\mathrm{L}}) \to (\mathcal{LZC}, g_{\mathrm{L}})$ *is a satellite operator of winding number* n*. Then* Π *is a bounded distance from the cabling operator* $C^m_{n,1}$*, where* m *is the unique number for which* $\mathrm{r}(\Pi \sqcup -C^m_{n,1}) = 0$.

Proof. The proof of [CH18, Proposition 6.3] considers the two pattern links associated to the two satellite operators, which are homologous in the solid torus. There, the key observation is that the minimum genus among all compact oriented surfaces cobounded by the two pattern links depends only on P. The proof then concludes by embedding the solid torus into S^3 as a tubular neighborhood of any given companion J.

We adapt this proof to the present context as follows. First, Π corresponds to a pattern Legendrian link $\Pi \subset J^1S^1$, with $[\Pi] = n \in H_1(J^1S^1)$, and the $C^m_{n,1}$-cabling operator corresponds to the pattern Legendrian link $C^m_{n,1} \subset J^1S^1$. By Corollary 3.14, Π is Lagrangian zigzag cobordant to $C^m_{n,1}$. One difference from the paragraph above is that, instead of one surface cobounded by Π and $C^m_{n,1}$, we now have an interpolating zigzag of Lagrangian surfaces. Specifically, there exist Legendrian links

$$\Pi = \Lambda_0, \Lambda_1, \ldots, \Lambda_{k-1}, \Lambda_k = C^m_{n,1}$$

and also Λ^+_i for $i = 1, \ldots, k$, together with Lagrangian cobordisms $L^<_i \subset \mathbb{R} \times J^1S^1$ from Λ_{i-1} to $\Lambda_{+,i}$, and cobordisms $L^>_i \subset \mathbb{R} \times J^1S^1$ from Λ_i to $\Lambda_{+,i}$. We observe here that the minimum total genus D among all choices of Lagrangian zigzag cobordisms also depends only on Π. The proof concludes now by embedding each $L^<_i, L^>_i \subset \mathbb{R} \times J^1S^1$ into $\mathbb{R} \times \mathbb{R}^3$ using Theorem 2.13, to obtain a Lagrangian zigzag cobordism of genus D between $\Pi(J)$ and $C^m_{n,1}(J)$. □

Corollary 3.16. *Any winding-number-*1 *satellite operator with rotation number* 0 *relative to the identity operator is a bounded distance from the identity operator. Any winding-number-*(-1) *satellite operator with rotation number* 0 *relative to the reverse operator is a bounded distance from the reverse operator.*

Proof. This follows from Proposition 3.15 and the definition of relative rotation numbers. □

Proof of Theorem 1.6. This is a direct consequence of Lemma 3.12 and Corollary 3.16. □

4. Maslov-0 Lagrangian zigzag cobordism

The key result in this section is Theorem 1.7, a Maslov-0 refinement of the main theorem of [SVW21] when the ambient contact manifold is the standard contact $\mathbb{R}^3$. The result is not only interesting in its own right, but also facilitates applications of Legendrian contact homology to questions of Lagrangian zigzag cobordisms, as we shall see in the next section.

The proof parallels that in [SVW21, Section 4]: we first find a Legendrian Λ_- and Lagrangian cobordisms L_- from Λ_- to Λ and L'_- from Λ_- to Λ'. We record these cobordisms using "handle graphs" Γ and Γ' on Λ_- (see [SVW21, Section 2.4] and below), and then attach the handles in both Γ and Γ' to Λ_- to create the desired Λ_+. The new step in this version of the proof is to take additional care to ensure that the handle graphs encode Maslov-0 Lagrangian cobordisms.

4.1. **Maslov-0 handle graphs.**

A Legendrian handle graph in the standard contact $\mathbb{R}^3$ is defined to be a pair (Γ, Λ), where Γ is a trivalent Legendrian graph and $\Lambda \subset \Gamma$ is a Legendrian link so that the vertices of Γ all lie on Λ and the set $\mathcal{H}$ of edges not in Λ is a finite collection of pairwise disjoint Legendrian arcs. As described in [SVW21], work of Dimitroglou Rizell [Dim16] or, in this case, Ekholm, Honda, and Kálmán [EHK16], implies that performing Legendrian ambient surgery on a subset $\mathcal{H}_0 \subset \mathcal{H}$ yields an exact Lagrangian cobordism $L(\Gamma, \Lambda, \mathcal{H}_0)$ from Λ to the Legendrian $\mathrm{Surg}(\Gamma, \Lambda, \mathcal{H}_0)$. If all edges $\mathcal{H}$ are used in the ambient surgery, we simply use the notation $\mathrm{Surg}(\Gamma, \Lambda)$. The goal of this section is to refine these ideas to keep track of Maslov indices.

We build on the notion of Maslov potentials for Legendrian graphs developed by [AB20] and exposited for front diagrams in [ABK22]. Denote the vertices of Γ by $\mathcal{V}$ and the cusps of the front diagram of Γ by $\mathcal{C}$. An m**-graded Maslov potential** on the front diagram of a Legendrian handle graph (Γ, Λ) is a function μ from the components of $\Gamma \setminus (\mathcal{V} \cup \mathcal{C})$ to $\mathbb{Z}/m$ that satisfies

$$\mu(u) = \mu(l) + 1$$

when the strands u and l meet at a cusp with the z values of l lower than those of u. See Figure 7 (a). It is straightforward to check that the front diagram of a Legendrian link whose components have vanishing rotation numbers has a 0-graded Maslov potential.

Definition 4.1. A **Maslov-m handle graph** is a handle graph (Γ, Λ) together with an m-graded Maslov potential μ on its front diagram such that, at each vertex of Γ, the values of μ of the adjacent edges are as in Figure 7 (b). We denote a Maslov-m handle graph by a triple (Γ, Λ, μ).

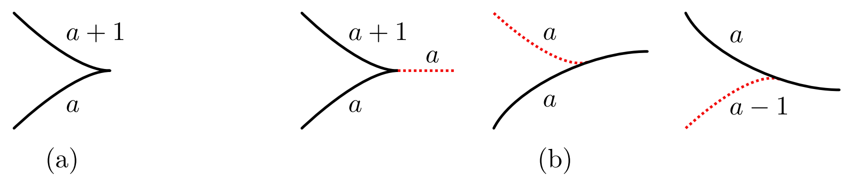

FIGURE 7. (a) A Maslov potential near a cusp. (b) A Maslov potential of the front diagram of a Maslov-m handle graph must satisfy three compatibility conditions at the vertices. The link Λ is indicated by solid arcs, while the handles are indicated by dotted arcs.

Lemma 4.2. *The property of being a Maslov-m handle graph is invariant under Legendrian isotopy.*

Proof. The proof is a straightforward, if somewhat tedious, case-by-case check of invariance under the six Reidemeister moves for Legendrian graphs in [OP12]. ☐

A vertex v of the front diagram of a handle graph (Γ, Λ) may be classified as either **smooth** if v is a smooth point of the diagram of Λ or **cusped** if v is a cusp of the diagram. We say that a front diagram of a handle graph has **flat handles** if all vertices are cusped and there are no cusps on the interiors of the handles; see Figure 8 for an example.

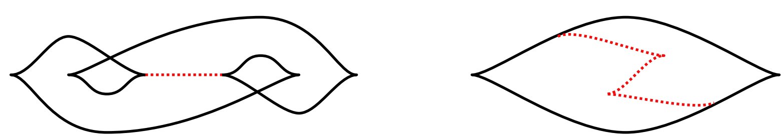

FIGURE 8. Legendrian handle graphs with (a) a flat handle and (b) a non-flat handle, both because the handle has cusps on the interior and because the vertices at its ends are smooth.

4.2. **Surgery on handle graphs.** We now have sufficient material to discuss surgery on handle graphs.

Lemma 4.3. *Any Maslov-m handle graph (Γ, Λ, μ) is Legendrian isotopic to a Maslov-m handle graph whose front diagram has flat handles.*

Proof. Convert every smooth vertex of Γ to a cusp vertex as in Figure 9 (a), then eliminate all cusps on handles as in Figure 9 (b). The result is a handle graph whose front diagram has flat handles. □

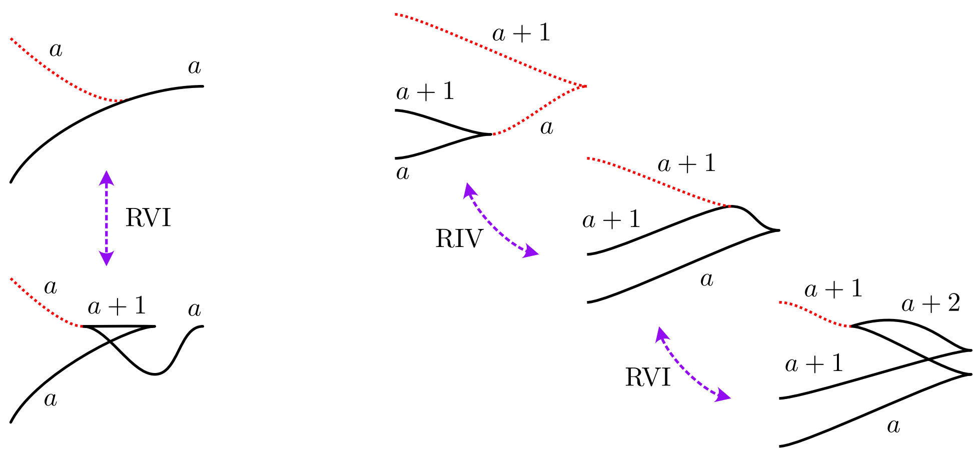

FIGURE 9. Converting a smooth vertex to a cusp vertex, left, and eliminating a cusp on a handle, right.

Lemma 4.4. *Legendrian ambient surgery on a Maslov-0 handle graph (Γ, Λ) yields a Maslov-0 Lagrangian cobordism from Λ to $\mathrm{Surg}(\Gamma, \Lambda)$.*

Proof. Lemma 4.3 allows us to assume that after an isotopy, all handles are flat. As shown in Figure 10, both Legendrian isotopy and attachment of a flat 1-handle generate fronts for Legendrian lifts of cobordisms with 0-graded Maslov potentials. It follows that any loop on the front diagram for L passes through the same number of cusps in an upward direction as it does in a downward direction. As in [RS20, Section 2.2], this shows that every loop has Maslov index 0, and hence that the cobordism L has Maslov number 0. □

Remark 4.5. A version of this result is well known to experts, namely that a decomposable Lagrangian cobordism has Maslov number 0 whenever (a) its 1-handles are attached along flat handles; and (b) these flat handles are between cusps whose incident strands have matching Maslov potentials. The purpose of the handle graph language introduced above is mainly to enable work with handle graphs without flat handles in the next subsection.

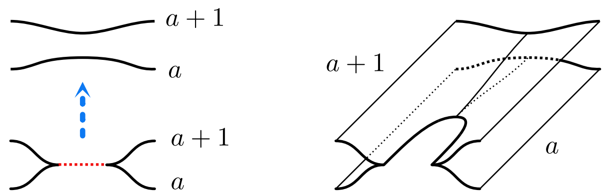

FIGURE 10. Surgery on a flat handle in a Maslov-0 handle graph yields a 0-graded Maslov potential on the front of the associated Lagrangian cobordism.

4.3. **Maslov-0 handle graphs for zigzag cobordisms.** The next steps in the proof of Theorem 1.7 parallel those in [SVW21, Section 4], making the diagrammatic operations in [SVW21, Lemmas 3.2, 4.2, 4.3, 4.4] that take a Legendrian Λ and produce a handle graph Γ on a stabilized unknot Λ_- with $\Lambda = \mathrm{Surg}(\Gamma, \Lambda_-)$ into Maslov-0 operations. This procedure involves the operation of pinching between two parallel strands of a front diagram to create a handle in a handle graph. We note that pinching two adjacent strands in a Maslov-m handle graph produces another Maslov-m handle graph if the Maslov potential of the higher strand is (modulo m) one more than that of the lower strand.

The necessary modifications to the lemmas in [SVW21] are described in the following two lemmas. The first of these constructs a Maslov-0 cobordism from a double-stabilization to the original link; this lemma replaces [SVW21, Lemma 3.2].

Lemma 4.6. *Given a Legendrian link Λ all of whose components have vanishing rotation number, there exists a Maslov-0 handle graph Γ on $S_{+-}(\Lambda)$ with $\Lambda = \mathrm{Surg}(\Gamma, S_{+-}(\Lambda))$.*

Proof. The proof is contained in Figure 11, with the condition that the rotation number is 0 yielding a 0-graded Maslov potential on Λ, and hence on $S_{+-}(\Lambda)$. □

The next lemma allows us to replace an orientable 1-handle attachment from Λ_- to Λ_+ with a Maslov-0 1-handle attachment from a stabilization of Λ_- to Λ_+. We say that an ambient surgery along a flat handle is **ordered** if the Maslov potential at the top of the co-core of the handle in Λ_+ is larger than the Maslov potential at the bottom.

Lemma 4.7. *Suppose the Legendrian links Λ_- and Λ_+ have components with vanishing rotation number. If there is a Lagrangian cobordism from Λ_- to Λ_+ induced by surgery on a single ordered 1-handle in a handle graph on Λ_-, then for some $k \geq 1$, there exists a Maslov-0 Lagrangian cobordism from $(S_{+-})^k(\Lambda_-)$*

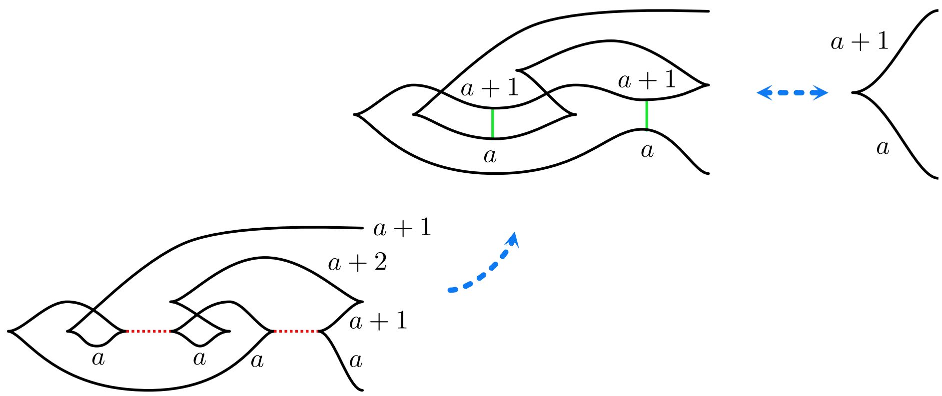

FIGURE 11. A handle graph on $S_{+-}(\Lambda)$ that yields Λ under Maslov-0 surgery.

to Λ_+ *induced by surgery on a Maslov-*0 *handle graph on* $(S_{+-})^k(\Lambda_-)$*. Further, the front for* $(S_{+-})^k(\Lambda_-)$ *has the same crossings as that of* Λ_-*.*

Proof. The proof is contained in Figure 12. Using a technique of [FT97], we may assume that, after a Legendrian isotopy, the front diagrams of $(S_{+-})^k(\Lambda_-)$ and Λ_- are identical outside a small neighborhood located in a place of our choosing in which the stabilizations are performed. In particular, after that isotopy, the front diagrams of $(S_{+-})^k(\Lambda_-)$ and Λ_- will have the same number of crossings. □

Proof of Theorem 1.7. As mentioned above, the proof follows the same steps as the proof in [SVW21, Section 4]. We reduce each of Λ and Λ' to Maslov-0 handle graphs on (stabilized) unknots using the following steps:

Elimination of negative crossings: A negative crossing is eliminated by a single pinch move as in [SVW21, Figure 17]; we may assume the pinch is ordered by choosing to pinch on the appropriate side of the crossing. Use Lemma 4.7 to replace the cobordism arising from the pinch with a Maslov-0 cobordism whose negative end has the same crossings as the original negative end.

Elimination of positive crossings: A leftmost positive crossing is eliminated by the more elaborate procedure depicted in [SVW21, Figure 18]. Only the last step needs to be adjusted, first by raising the Maslov potential of the top strand of the final two pinches using the same combination of Reidemeister-I moves and pinches in earlier stages of the procedure, and then by replacing the final two pinches using Lemma 4.7.

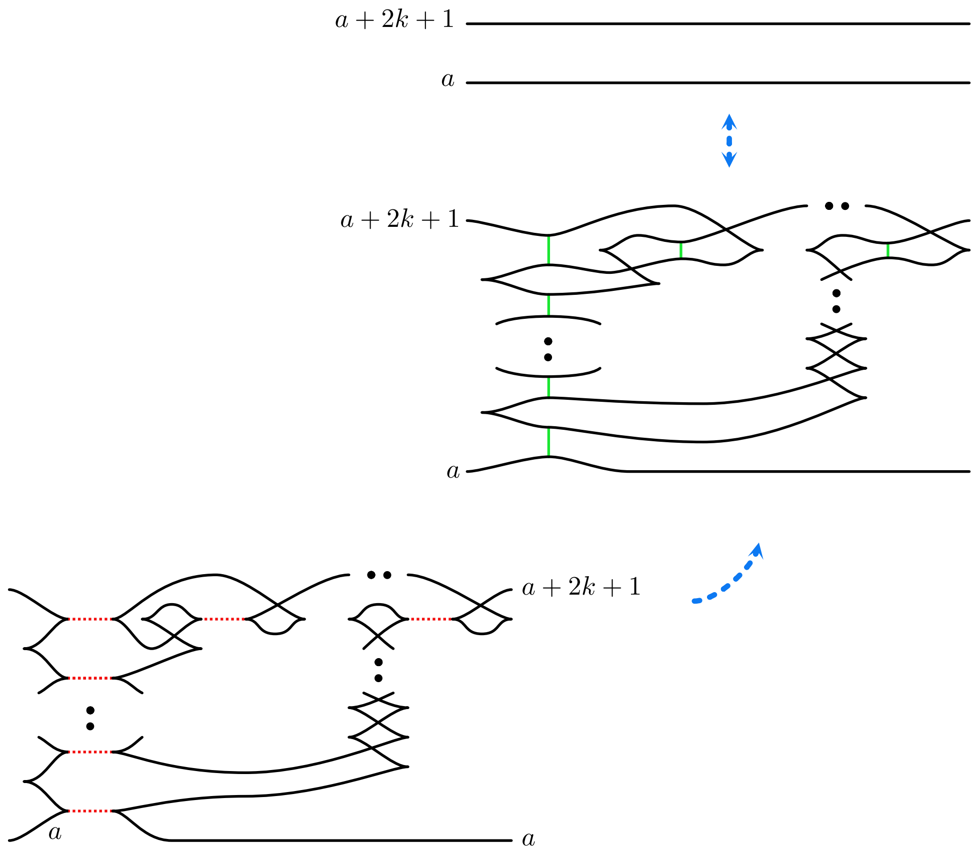

FIGURE 12. A handle graph on $S^k_{+-}(\Lambda_-)$ that yields Λ_+ under a Maslov-0 surgery, where Λ_+ is obtained from Λ_- by a single orientable 1-handle attachment.

Merging of components: This step may be adjusted by simply adding appropriate Reidemeister-I moves to the procedure in [SVW21, Figure 19] to ensure that the pinches are all Maslov-0 operations.

The unknots are then stabilized further using Lemma 4.6 until they have the same Thurston–Bennequin numbers (they already have rotation number 0 by construction). The resulting unknots are Legendrian isotopic; we call the result Λ_-. By dragging the handle graphs along with the isotopy and taking a union of the handles, we obtain (Γ_-, Λ_-). By Lemma 4.4, the partial surgery procedure in [SVW21] results in the desired Maslov-0 Lagrangian cobordisms. □

5. Relationship to non-classical invariants

In this section, we discuss the relationship between non-classical invariants and Lagrangian zigzag cobordism. The section begins with a review of Legendrian contact homology in Section 5.1, proceeds to apply it to zigzag cobordisms in Section 5.2, and ends with a proof of a setwise Poincaré–Chekanov polynomial geography theorem (Theorem 1.8).

5.1. **Background on Legendrian contact homology.** Lagrangian cobordisms are obstructed by the Legendrian contact homology differential graded algebra (LCH DGA), whose homology is a non-classical invariant of Legendrian knots. In this subsection, we briefly set notation and review key results. See the survey [EN22] for a more comprehensive introduction.

The LCH DGA $(\mathcal{A}_\Lambda, \partial_\Lambda)$ of the Legendrian knot Λ in the standard contact $\mathbb{R}^3$ was first defined by Chekanov [Che02]; see also [Eli98]. The algebra (over the field $\mathbb{F}_2$, for simplicity) is generated by the Reeb chords of Λ and graded by a Conley–Zehnder index. The differential counts certain immersed disks in $\mathbb{R}^2$ with boundary on the Lagrangian projection of Λ; these disks correspond to holomorphic disks in the symplectization $\mathbb{R} \times \mathbb{R}^3$ [EGH00, ENS02].

While it can be difficult to distinguish knots from presentations of their DGAs, Chekanov's linearization procedure yields a more computable set of invariants. Given a DGA $(\mathcal{A}_\Lambda, \partial_\Lambda)$, its **augmentations** are DGA maps $\varepsilon\colon (A_\Lambda, \partial_\Lambda) \to (\mathbb{F}_2, 0)$. Each augmentation ε induces a differential $\partial_\Lambda^\varepsilon$ on the $\mathbb{F}_2$ vector space A_Λ generated by the Reeb chords of Λ. The homology of the chain complex $(A_\Lambda, \partial_\Lambda^\varepsilon)$ is called the **linearized Legendrian contact homology** $\mathrm{LCH}_*(\Lambda, \varepsilon)$ **with respect to** ε. It is convenient to record the dimensions of $\mathrm{LCH}_*(\Lambda, \varepsilon)$ in a **Poincaré–Chekanov polynomial**

$$p_{\Lambda,\varepsilon}(t) = \sum_{i=-\infty}^{\infty} \dim \mathrm{LCH}_i(\Lambda, \varepsilon)\, t^i.$$

A duality result for Legendrian contact homology [Sab06] implies that every Poincaré–Chekanov polynomial is of the form

$$p_{\Lambda,\varepsilon}(t) = t + q(t) + q(t^{-1}) \tag{5.1}$$

for some polynomial q. The set $\mathcal{P}_\Lambda$ of Poincaré–Chekanov polynomials over all possible augmentations is an invariant of the Legendrian knot Λ. For example, Chekanov used this fact to show that the twist knots Λ and Λ' in Figure 13 are not Legendrian isotopic despite having the same classical invariants since $\mathcal{P}_\Lambda = \{2 + t\}$ while $\mathcal{P}_{\Lambda'} = \{t^{-2} + t + t^2\}$ [Che02].

As hinted above, Legendrian contact homology is functorial under Lagrangian cobordism. More precisely, an exact, Maslov-0 Lagrangian cobordism L from Λ_- to Λ_+ induces a DGA map $\phi_L : (\mathcal{A}_{\Lambda_+}, \partial_{\Lambda_+}) \to (\mathcal{A}_{\Lambda_-}, \partial_{\Lambda_-})$ [EHK16].

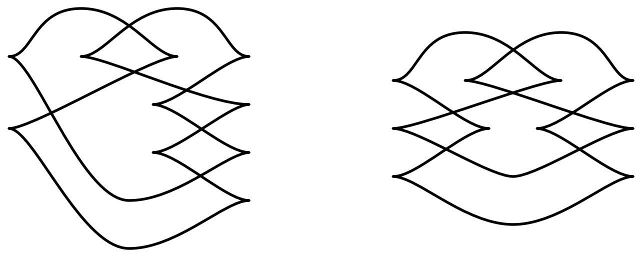

FIGURE 13. Two non-isotopic Legendrian representatives of $m(5_2)$.

Such a DGA map can be used to pull back augmentations for Λ_- to augmentations for Λ_+ via $\phi_L^*\varepsilon_- = \varepsilon_- \circ \phi_L$, and hence linearized Legendrian contact homology can be used to obstruct Lagrangian cobordisms. We recall this obstruction from Pan [Pan17]:

Theorem 5.2 ([Pan17, Corollary 1.4])**.** *If there is an exact, Maslov-*0 *Lagrangian cobordism L from Λ_- to Λ_+, and if ϵ_- is an augmentation of $(\mathcal{A}_{\Lambda_-}, \partial_{\Lambda_-})$, then*

$$\mathrm{LCH}_*(\Lambda_+, \phi_L^*\varepsilon_-) \cong \mathrm{LCH}_*(\Lambda_-, \varepsilon_-) \oplus \mathbb{F}^{-\chi(L)}[0],$$

where $\mathbb{F}^{-\chi(L)}[0]$ denotes the vector space $\mathbb{F}^{-\chi(L)}$ in degree 0. In particular, if L is a concordance, then the associated linearized Legendrian contact homologies are isomorphic.

5.2. **Linearized LCH and zigzag cobordisms.** Theorem 5.2 shows that if there is a Lagrangian concordance L from Λ_- to Λ_+, then $\mathcal{P}_{\Lambda_-} \subset \mathcal{P}_{\Lambda_+}$. This shows, for example, that not only are the twist knots in Figure 13 not Legendrian isotopic, but there is no Lagrangian concordance between them in either direction.

Even so, the structure of the linearized LCH invariant does not yield an obstruction to Maslov-0 zigzag cobordism. To see why, suppose that $\mathbf{L}(\Lambda_1, \Lambda_2)$ is a Lagrangian cospan with top Legendrian Λ_+. While the sets of augmentations of Λ_i, $i = 1, 2$, both pull back to augmentations of Λ_+, these sets may be disjoint. In particular, we might have that the subsets $\mathcal{P}_{\Lambda_1}, \mathcal{P}_{\Lambda_2} \subset \mathcal{P}_{\Lambda_+}$ are disjoint, and hence we cannot fruitfully use comparisons between the sets $\mathcal{P}_{\Lambda_i}$ to obstruct a Lagrangian zigzag cobordism.

Remark 5.3. This issue of not being able to guarantee that the sets of pulled-back invariants match up—and hence being unable to use the sets of invariants to obstruct Lagrangian zigzag cobordisms—is common to linearized LCH, generating family homology [ST13, Tra01], and sheaf invariants [Li22, STZ17].

The use of Legendrian invariants $\widehat{\mathcal{L}}(\Lambda)$ in knot Heegaard Floer homology [BLW22, GJ19] or knot monopole Floer homology [BS18] suffers from an issue that is perhaps similar in spirit, but differently manifested. In, say, Heegaard Floer theory, one has homomorphisms $\mathcal{F}_i\colon \widehat{\mathrm{HFK}}(\Lambda_+) \to \widehat{\mathrm{HFK}}(\Lambda_i)$ such that $\mathcal{F}_i(\widehat{\mathcal{L}}(\Lambda_+)) = \widehat{\mathcal{L}}(\Lambda_i)$. But knowing only $\widehat{\mathcal{L}}(\Lambda_i)$—even if one vanishes and the other does not—one is never able to disprove the existence of *some* unknown, non-vanishing $\widehat{\mathcal{L}}(\Lambda_+)$ and corresponding homomorphisms $\mathcal{F}_i$ satisfying the above, which would be necessary to obstruct all possible zigzag cobordisms between Λ_1 and Λ_2.

Open Question 5.4. Figure 13 showcases an instance of the Legendrian botany question: How many different Legendrian knots (up to isotopy) are there for a fixed set of classical invariants? Since the classical invariants are preserved by Lagrangian zigzag concordance, it is reasonable to ask whether the two knots in Figure 13 are zigzag concordant. Pushing further, one could restrict g_{L} to the set of Legendrians with common smooth knot class and classical invariants to *quantify* the botany question: What is the diameter of such a set of Legendrian knots? An effective nonclassical invariant of Lagrangian zigzag concordance seems necessary to answer this question.

One possible way to deal with the structural failure of linearized LCH to yield an obstruction to zigzag cobordism is as follows: One could refine the Lagrangian zigzag cobordism relation to take pairs (Λ, ε) of Legendrians and augmentations as its objects, while insisting on matching pullbacks of augmentations at the top in a Lagrangian cospan. More precisely, an **augmented Lagrangian cospan** $\mathbf{L}(\varepsilon, \varepsilon')$ consists of an augmented Legendrian link $(\Lambda_+, \varepsilon_+)$ and two exact, connected Maslov-0 Lagrangian cobordisms L from Λ to Λ_+ and L' from Λ' to Λ_+ so that $\varepsilon \circ \phi_L = \varepsilon_+ = \varepsilon' \circ \phi_{L'}$; the definitions of the augmented zigzag cobordism relation $\overset{\varepsilon}{\sim}$ and concordance relation $\overset{\varepsilon}{\approx}$ then follow the same structure as Definition 2.3. Note that Theorem 5.2 implies that linearized LCH is an invariant of augmented zigzag concordance and that components of linearized LCH in nonzero grading are invariants of augmented zigzag cobordism. This refinement shifts the study of Lagrangian zigzag concordance from Legendrians to augmentations. For example, we prove the following:

Proposition 5.5. *For any $n \in \mathbb{N}$, there exists a Legendrian Λ_n with augmentations $\varepsilon_1, \dots \varepsilon_n$ so that $(\Lambda_n, \varepsilon_i) \overset{\varepsilon}{\not\sim} (\Lambda_n, \varepsilon_j)$ for $i \neq j$.*

Proof. Following the proof of [Siv11, Corollary 5.2], let Λ_0 be the maximal Legendrian unknot, and let Λ_n be the Legendrian (i.e. tb-twisted) Whitehead double of Λ_{n-1}. Sivek showed that Λ_n has augmentations $\{\varepsilon_1, \dots, \varepsilon_n\}$ with Poincaré-Chekanov polynomials $\{p_1, \dots, p_n\}$ that satisfy $p_1(t) = t + 2$ and

$$p_k(t) = t + 2 + (t + 2 + t^{-1})(p_{k-1}(t) - t). \tag{5.6}$$

Using (5.6), we may inductively compute that, for $n \geq 2$, $\deg p_n = n - 1$. The proof now follows from Theorem 5.2, as noted above. □

In a different direction, we may also *exploit* the interaction between lineared LCH and a Lagrangian cospan, not to obstruct zigzag cobordisms, but to prove other interesting statements. In particular, we may use the fact that the Legendrian Λ_+ at the top of a Lagrangian cospan $\mathbf{L}(\Lambda_1, \Lambda_2)$ pulls back linearized LCH information from both Λ_1 and Λ_2 to construct Legendrian knots with (almost) arbitrary sets of Poincaré–Chekanov polynomials. The procedure is made precise by Theorem 1.8, which tells us that for any set of Laurent polynomials compatible with duality as in (5.1), there is a Legendrian knot that contains those polynomials—up to a correction in grading 0—in its set of Poincaré–Chekanov polynomials.

Proof of Theorem 1.8. By [MS05, Theorem 1.2], for each $i \in \{1, \dots, n\}$, there exists a Legendrian knot Λ_i and an augmentation ε_i of its LCH DGA so that $p_{\Lambda_i,\varepsilon_i}(t) = p_i(t) + p_i(t^{-1}) + t$. Theorem 1.7 and induction yield a Legendrian knot Λ_+ and Maslov-0 Lagrangian cobordisms $\Lambda_i \prec_{L_i} \Lambda_+$. Let $c_i = \chi(L_i)$. Finally, Theorem 5.2 gives augmentations $\varepsilon_{+,i}$ of Λ_+ so that

$$p_{\Lambda_+,\varepsilon_{+,i}}(t) = p_{\Lambda_i,\varepsilon_i}(t) + \chi(L_i) = p_i(t) + p_i(t^{-1}) + t + c_i,$$

which proves the theorem. □

Remark 5.7. The constants c_i are not independent. In fact, once c_1 is known, the other constants are determined. The idea is that the formula $\operatorname{tb}(\Lambda_+) = p_{\Lambda_i,\varepsilon_i}(-1) + c_i$ holds for any i. Knowing c_1 determines $\operatorname{tb}(\Lambda_+)$, and then, in turn, $\operatorname{tb}(\Lambda_+)$ determines $c_2, \dots, c_n$.

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