① Sphere and Disk Theorems

Let's begin by recalling some basic facts from algebraic topology.

Coverings: Suppose $p: \tilde{X} \to X$ is a covering map

- Then $p_*: \pi_i(\tilde{X}) \to \pi_i(X)$ is an isomorphism for all $i \geq 2$.
- If $f: Y \to X$ satisfies $f_*(\pi_1(Y)) \subset p_*(\pi_1(\tilde{X}))$, then $\exists$ a lift

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ Y \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

Hurewicz: Suppose that X is connected. Then there is a Hurewicz map
$$h_k: \pi_k(X) \longrightarrow H_k(X)$$
When $k=1$, this is the usual abelianization.

The Hurewicz Theorem asserts that if $\pi_1(X) = 1$ and $n \geq 2$, then
$$\underset{2 \leq k < n}{\pi_k(X) = 0} \iff \underset{2 \leq k < n}{H_k(X) = 0}$$

and $h_n: \pi_n(X) \to H_n(X)$ is an isomorphism whenever the above is true.

Whitehead: Let X + Y be connected CW complexes + $f: X \to Y$ a cont. map.

- If $f_*: \pi_i(X) \to \pi_i(Y)$ is an isom. for all i, then f is a homotopy equiv.
- If $\pi_1(X) = \pi_1(Y) = 1$ + $f_*: H_i(X) \to H_i(Y)$ is an isom for all i, then f is a homotopy equiv.

Df^{n} We say that X is a $K(\pi, n)$ if $\pi_n(X) = \pi$ and $\pi_i(X) = 0$ for all $i \neq n$.
If X is a $K(\pi, 1)$, then we say that X is aspherical

②

Poincaré-Lefshetz Duality Suppose that M is a compact, oriented n-manifold. Then the following diagram commutes, with all vertical maps isomorphisms:

$$\begin{tikzcd}
\cdots \arrow[r] & H^p(M,\partial M) \arrow[r] \arrow[d] & H^p(M) \arrow[r] \arrow[d] & H^p(\partial M) \arrow[r] \arrow[d] & H^{p+1}(M,\partial M) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H_{n-p}(M) \arrow[r] & H_{n-p}(M,\partial M) \arrow[r] & H_{n-p-1}(\partial M) \arrow[r] & H_{n-p-1}(M) \arrow[r] & \cdots
\end{tikzcd}$$

Universal Coefficients Theorem For any pair (X,A),

$$H^n(X,A;\mathbb{Z}) \cong \underbrace{\mathrm{Free}(H_n(X,A;\mathbb{Z}))}_{\text{Given by } \mathrm{Hom}(H_n(X,A;\mathbb{Z}),\mathbb{Z})} \oplus \mathrm{Tor}(H_{n-1}(X,A;\mathbb{Z}))$$

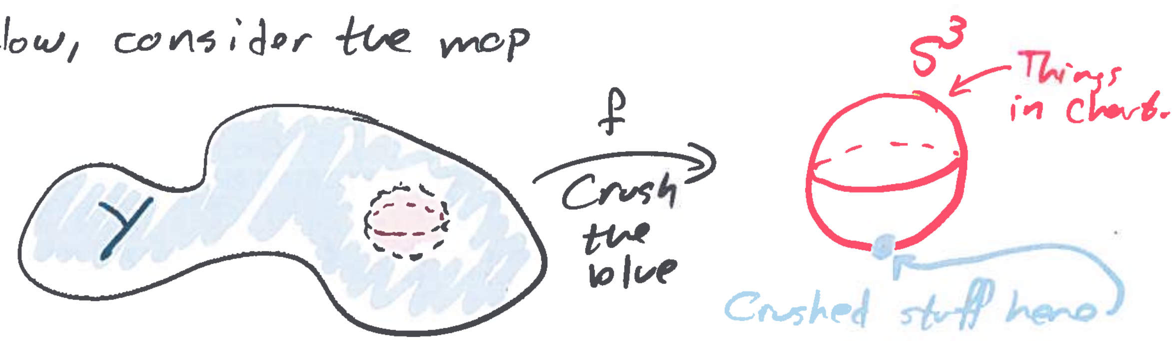
Lemma Ⓐ Let Y be a closed, connected 3-mfd. Then $\pi_1(Y)=1$ iff $Y \simeq S^3$.

proof: Since $\pi_1(Y)=1$, $H_1(Y)=0$

Also,
$$H_2(Y) \overset{PD}{\cong} H^1(Y) = \mathrm{Free}(H_1(Y)) \oplus \mathrm{Tor}(H_0(Y)) = 0$$

Finally,
$$H_3(Y) \cong \mathbb{Z}.$$

Now, consider the map



This map clearly has degree $=1$, thus,
$$f_*: H_i(Y) \longrightarrow H_i(S^3)$$
is an isomorphism $\forall i$.

By Whitehead, f is a homotopy equiliv ☒

Lemma B Suppose that Y is non-compact and has $\partial Y = \emptyset$. Then
$$\pi_1(Y) = \pi_2(Y) = 0 \text{ iff } Y \simeq \mathbb{R}^3.$$

Proof: Since Y is non-compact, $H_3(Y) = 0$.

By Hurewicz, we have that $\pi_i(Y) = 0 \ \forall i \geq 1$.

Thus, the constant map
$$f: Y \longrightarrow \mathbb{R}^3$$
is a homotopy equivalence. $\blacksquare$

We're now ready to state

Theorem (Sphere Theorem, Papakyriakopoulos '57, Whitehead '58) Let Y be a 3-manifold, $f: S^2 \longrightarrow Y$ a map with $[f] \neq 0$ as an element of $\pi_2(Y)$. Then $\exists$ an embedding $e: S^2 \hookrightarrow Y$ with $[e] \neq 0$ in $\pi_2(Y)$.

Theorem (Disk Theorem, Papakyriakopoulos '57) Let Y be a 3-manifold and Σ a subsurface of ∂Y. Suppose $f: (D, \partial D) \longrightarrow (Y, \Sigma)$ is a map with $[f|_\partial] \neq 0 \in \pi_1(\Sigma)$. Then $\exists$ an embedding $e: (D, \partial D) \longrightarrow (Y, \Sigma)$ with $[e|_{\partial D}] \neq 0$

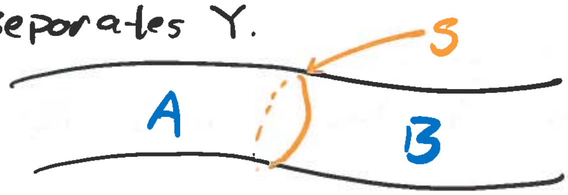
Corollary 4 Let Y be a 3-manifold. Then
$$Y \text{ irreducible} \implies \pi_2(Y) = 0$$
$$\pi_2(Y) \implies Y \text{ irreducible (Given PC)}.$$

Proof ($\Rightarrow$) Suppose that $\pi_2(Y) \neq 0$. By the sphere theorem, $\exists$ an embedding $e: S^2 \longrightarrow Y$ with
$$[e] \neq 0 \text{ in } \pi_2(Y).$$

In particular, this sphere does not bound a 3-ball in Y.

($\Leftarrow$) Let $S \subset Y$ be an embedded 2-sphere. Then
$$[S] = 0 \in \pi_2(Y) \implies [S] = 0 \in H_2(Y).$$

Thus, S separates Y.



④ Let $p: \tilde{Y} \to Y$ be the universal cover of Y.
Then $p^{-1}(A) = \coprod \tilde{A}$ and $p^{-1}(B) = \coprod \tilde{B}$ are disjoint unions of a bunch of copies of the universal covers of A and B respectively.
(this is true because S is a 2-sphere).

In this case,
$$\partial \tilde{A} = \coprod_{|\pi_1(A)|} S \quad , \quad \partial \tilde{B} = \coprod_{|\pi_1(B)|} S .$$

Suppose that S_0 is a lift of S.
Since $\pi_2(Y) = 0$, $\pi_2(\tilde{Y}) = 0$ as well.
By Hurewicz, $\pi_1(\tilde{Y}) = \pi_2(\tilde{Y}) = 0 \implies H_2(\tilde{Y}) = 0$.
$$\text{Thus, } [S_0] = 0 \in H_2(\tilde{Y}).$$

<u>Claim:</u> Let Y be a 3-manifold. Suppose Σ is a component of ∂Y. If $[\Sigma] = 0 \in H_2(Y)$, then Y is compact and $\partial Y = \Sigma$.

We have that
$$\tilde{Y} = X \cup_{S_0} Z, \quad \text{with } S_0 \text{ separating.}$$

Mayer-Vietoris $\implies$
$$\underset{\mathbb{Z}}{\overset{\text{\rotatebox{90}{$\cong$}}}{H_2(S_0)}} \to H_2(X) \oplus H_2(Z) \to \underset{0}{\overset{}{H_2(\tilde{Y})}} \implies H_2(X) \text{ or } H_2(Z) \text{ (or both)} = 0.$$

Thus, by the claim, we can assume WLOG, X is compact + $\partial X = S_0$.
This implies that $X = \tilde{A}$ or $X = \tilde{B}$. Assume WLOG, that $X = \tilde{A}$. Then
$$\partial \tilde{A} = S_0$$
Since $|\partial \tilde{A}| = 1$, we have that $\pi_1(A) = 1$. In particular,
$$\tilde{A} \cong A.$$
Thus, $A \cup_\partial B^3$ is closed and has $\pi_1 = 1$. By PC, $A \cong$ (3-ball)

⑤

Now, to prove the claim

Since $[\Sigma]=0$ in $H_2(Y)$, we have a compact submanifold $Y_0 \subset Y$ with $[\Sigma]=0$ in $H_2(Y_0)$.

Thus, we can assume Y is compact and need to show that
$$\partial Y = \Sigma.$$
Suppose not. Then

$$\begin{tikzcd} H_3(Y) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & H_3(Y,\partial Y) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & H_2(\partial Y) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & H_2(Y) \\ 0 & H^0(Y) \arrow[r, "i^*"] \arrow[d, phantom, "\cong" rotate=90] & H^0(\partial Y) \arrow[d, phantom, "\cong" rotate=90] & \\ & \mathbb{Z} & \mathbb{Z}\oplus\cdots\oplus\mathbb{Z} & \end{tikzcd}$$

Where i^* is the Hom dual of

$$\begin{array}{ccc} H_0(\partial Y) & \longrightarrow & H_0(Y) \\ \cong & & \cong \\ \mathbb{Z}\oplus\cdots\oplus\mathbb{Z} & & \mathbb{Z} \\ (0,\ldots,1,\ldots,0) & \longmapsto & 1 \end{array}$$

In particular, $i^*(1)=(1,\ldots,1)$ and $[\Sigma] \notin \operatorname{Im}(i^*)$ unless $\partial Y = \Sigma$.

$\boxed{/\!/}$

Theorem: Suppose that Y is a closed 3-manifold with universal cover $\tilde{Y}$.
- If $\pi_1(Y)$ is finite, then $\tilde{Y} \cong S^3$.

  If $\pi_1(Y)$ is infinite + Y is prime then either $\tilde{Y} \cong \mathbb{R}^3$ or $Y \cong S^1 \times S^2$