Complex Analysis Homework 7

Problem 1: (Type 1 problem) Suppose that f and g are entire holomorphic functions such that

$$|f(z)| \leq |g(z)|$$

for all $z \in \mathbb{C}$. Prove that there exists a constant c such that $f(z) = c \cdot g(z)$ for all $z \in \mathbb{C}$.

Problem 2: Let f be non-constant and holomorphic in an open set containing the closed unit disk.

(a) Show that if $|f(z)| = 1$ whenever $|z| = 1$, then the image of f contains the unit disk. [Hint: One must show that $f(z) = w_0$ has a root for every $w_0 \in \mathbb{D}$. To do this, it suffices to show that $f(z) = 0$ has a root (why?). Use the maximum modulus principle to conclude.]

(b) If $|f(z)| \geq 1$ whenever $|z| = 1$ and there exists a point $z_0 \in \mathbb{D}$ such that $|f(z_0)| < 1$, then the image of f contains the unit disk.

Problem 3: Prove that if f is an entire function with the property that, for all $z \in \mathbb{C}$

$$|f(z)| \leq A|z|^N + B$$

for some integer $N \geq 0$ and some constants $A, B > 0$, then f is a polynomial of degree $\leq N$. [Hint: Use a Liouville type argument for the $(N+1)^{st}$ order derivative $f^{(N+1)}$.]

Problem 4: (Pledged) Show that if the real part of an entire function f is bounded, then f is constant.