Complex Analysis – Final Assignment 2023

You must work independently on each of these questions, but are ENTHUSIASTICALLY encouraged to meet with me in office hours or otherwise to discuss progress or to have any quesitons answered!

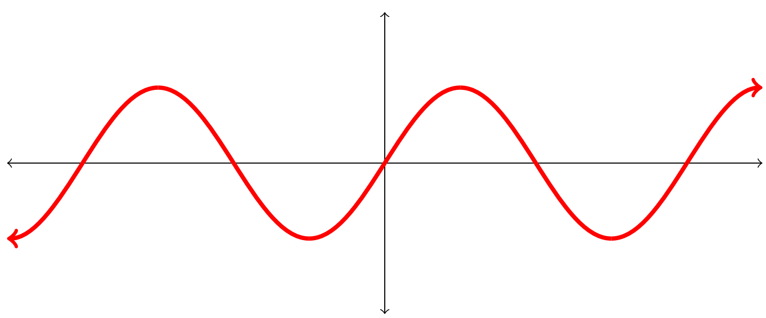
Question 1: What does it means for a function to be *holomorphic* on a domain $D \subset \mathbb{C}$? What does it mean for a function to be *conformal* at a point $z \in D$. How are these two properties related?

Question 2: Compute the following principle value integral.

$$PV \int_{-\infty}^{\infty} \frac{1}{x^4 - 1}\, dx.$$

Question 3: True or False: There exists a holomorphic function $f(z)$ on the unit disk $D = \{z \in \mathbb{C} \,|\, |z| \leq 1\}$ for which $f(0) = 3$ and $f(z)$ restricted to the boundary of D is the identity (i.e. $f(z) = z$ for each $z \in S^1 = \partial D$).

Question 4: Consider the curve γ in $\mathbb{C}$ shown below (i.e., the graph of $\sin(x)$). Prove that γ cannot possibly be the image of a holomorphic function $f : \mathbb{C} \to \mathbb{C}$.



Question 5: Give an example of a Möbius transformation $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ for which

$$f(1) = 0$$
$$f(0) = \infty$$
$$f(-1) = 1.$$

Also, give an explicit formula for $f^{-1}(z)$ as a Möbius transformation?

Question 6: Let $a \in \mathbb{C}$ be a complex number, and consider the function

$$f(z) = \frac{e^{1/z}}{z+a}$$

Compute the residues of $f(z)$ at each of its singularities.

Question 7: True or False: There exists an entire holomorphic function $f(z)$ satisfying $f(0) = 8$ and $|f(z)| \leq 1/R$ for $|z| = R >> 0$?