Complex Analysis Homework 6

Problem 1: (Type 1 problem) Prove that all entire functions that are also injective take the form $f(z) = az + b$ with $a, b \in \mathbb{C}$, and $a \neq 0$. [Hint: Apply the Casorati-Weierstrass Theorem to $f(1/z)$.]

Problem 2: The function $\cos(1/z)$ has an essential singularity at 0. Verify the truth of the Great Picard Theorem directly for this function.

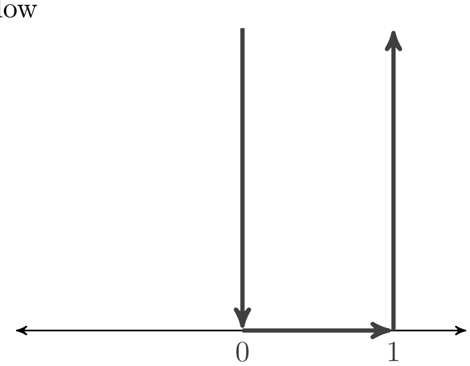
Problem 3: Show that for $n \geq 1$,

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^{n+1}} = \frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{2 \cdot 4 \cdot 6 \ldots (2n)} \cdot \pi.$$

Problem 4: Show that

$$\int_0^1 \log(\sin(\pi x))\, dx = -\log(2).$$

Hint: Use the contour below



Problem 5: (Type 1 problem) Suppose that f has a pole of at z_0 of order $\leq N$. The function $(z - z_0)^N f(z)$ has a removable singularity at z_0. Prove that

$$\mathrm{Res}(f, z_0) = \left. \frac{\left(\frac{d}{dz}\right)^{N-1} \left[(z - z_0)^N f(z)\right]}{(N-1)!} \right|_{z=z_0}$$