Complex Analysis Homework 5

Problem 1: (Pledged) Let Ω be an open subset of $\mathbb{C}$ and let $C \subset \Omega$ be a circle whose interior is also contained in Ω. Suppose that f is a function which is holomorphic on Ω except possibly at a point w inside of C. Prove that if f is bounded near w, then

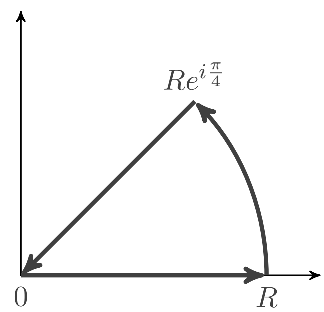
$$\int_C f(z)\,dz = 0.$$

Problem 2: Show that

$$\int_0^\infty \sin(x^2)\,dx = \int_0^\infty \cos(x^2)\,dx = \frac{\sqrt{2\pi}}{2}.$$

These are the **Fresnel integrals**. Here, $\int_0^\infty$ is interpreted as $\lim_{R\to\infty}\int_0^R$.

[Hint: Integrate the function e^{-z^2} over the path below and recall that $\int_{-\infty}^{\infty} e^{-x^2} = \sqrt{\pi}$.]



Problem 3: Suppose f is continuous in a region Ω. Show that, if they exist, any two primitives of f (i.e., functions $F(z)$ such that $F'(z) = f(z)$) differ by a constant.

Problem 4: Let $\gamma : [0,1] \to \mathbb{C}$ be a smooth curve and $-\gamma$ the curve with the same image as γ, but traversed in the reverse direction (i.e., $-\gamma(t) = \gamma(1-t)$). Show that for any continuous function f defined along γ,

$$\int_\gamma f(z)\,dz = -\int_{-\gamma} f(z)\,dz.$$

Bonus Problem: Suppose that f is an entire function on $\mathbb{C}$ with the property that for each $z_0 \in \mathbb{C}$ at least one coefficient of the power series expansion

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

is equal to 0. Prove that f is a polynomial.

[Hint: Use the fact that $c_n n! = f^{(n)}(z_0)$ together with a countability argument.]