Complex Analysis Homework 4

Recall that there is a unique Möbius transformation f of $\widehat{\mathbb{C}}$ which satisfies

$$\begin{aligned} f(0) &= -1, \\ f(\infty) &= 1, \\ f(i) &= 0. \end{aligned}$$

This Möbius function is called the *Caley transformation.*

Problem 1: Write explicitly

$$f(z) = \frac{az+b}{cz+d}.$$

That is, find a, b, c, d.

Problem 2: (Type 1 problem) Give an elementary (but still rigorous) explanation of the fact that

$$f(\mathbb{R} \cup \{\infty\}) = \{\text{the unit circle in } \mathbb{C}\}.$$

Problem 3: (Type 1 problem) Explain why the image under f of the open upper half-plane is the open unit disk $\mathbb{D} \subset \mathbb{C}$.

Problem 4: (Type 1 problem) For several values of $y > 0$, sketch the image of the straight lines in the upper half plane.

$$\{f(x+iy) \mid x \in \mathbb{R} \cup \infty\}.$$

Absent explicit formulas, give an "intuitive" explanation of why your pictures make sense.

Problem 5: Consider the function defined on $\mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ e^{-1/x^2} & \text{if } x > 0. \end{cases}$$

Show that f is infinitely differentiable on $\mathbb{R}$, and that $f^{(n)}(0) = 0$ for all $n \geq 1$. Conclude that f does not have a converging power series expansion $\sum_{n=0}^{\infty} a_n x^n$ for x near the origin.

Problem 6: (Pledged) Give examples of power series $f(z)$ and $g(z)$ so that

- $f(z)$ has radius of convergence 1,
- $g(z)$ has radius of convergence 2,
- $f(z) \cdot g(z)$ has radius of convergence 10.