Complex Analysis Homework 3

Problem 1: Recall that if a complex valued function $f : D \to \mathbb{C}$ is holomorphic, then it satisfies the **Cauchy-Riemann equation**

$$\frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y}.$$

This motivates us to define two differential operators

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right) \qquad \text{and} \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right).$$

Let's observe two facts. First, a function f satisfies Cauchy-Riemann if and only if $\frac{\partial f}{\partial \bar{z}} = 0$. Second, if a function is indeed complex differentiable, then $\frac{df}{dz} = \frac{\partial f}{\partial z}$.
Show that

$$4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} = 4\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z} = \Delta,$$

where Δ is the **Laplacian**

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

Problem 2: A function $h : \mathbb{R}^2 \to \mathbb{R}$ is said to be **harmonic** if $\Delta h = 0$. Use what you learned from Problem 1 to prove that if f is holomorphic on the open set $\Omega \subset \mathbb{C}$, then the real and imaginary parts of f are harmonic.