Complex Analysis Homework 1

Problem 1: Describe geometrically the sets of points z in the complex plane defined by the following relations:

(a) $|z - z_1| = |z - z_2|$
(b) $1/z = \bar{z}$
(c) $Re(z) = 3$
(d) $Re(z) > c$ (resp., $\geq c$) where $c \in \mathbb{R}$
(e) $Re(az + b) > 0$ where $a, b \in \mathbb{C}$
(f) $|z| = Re(z) + 1$
(g) $Im(z) = c$ with $c \in \mathbb{R}$

Problem 2: Let $\langle \cdot, \cdot \rangle$ denote the usual inner product (dot product) in $\mathbb{R}^2$. In other words, if $Z = (x_1, y_1)$ and $W = (x_2, y_2)$, then

$$\langle Z, W \rangle = x_1 x_2 + y_1 y_2.$$

Similarly, we may define a Hermitian inner product in $\mathbb{C}$ by

$$(z, w) = z\bar{w}.$$

The term Hermitian is used to describe the fact that $(\cdot, \cdot)$ is not symmetric, but rather satisfies the relation

$$(z, w) = \overline{(w, z)} \quad \text{for all}\, z, w \in \mathbb{C}.$$

Show that

$$\langle z, w \rangle = \frac{1}{2}[(z, w) + (w, z)] = Re(z, w)$$

where we use the usual identification $z = x + iy \in \mathbb{C}$ with $(x, y) \in \mathbb{R}^2$.

Problem 3: [*The following problem is from Prof. Jones's notes, which also contain a detailed solution. Please do not look at the solution from the notes until after you have complected (and written up!) the problem yourself.*]

The goal of this problem is to understand the "geometry" of the complex function $1/z$. Let C be a circle in $\mathbb{C}$ with center $a \in \mathbb{C}$ and radius $r > 0$. In class, we showed that a complex number z is in C if and only if

$$|z|^2 - 2\mathrm{Re}(z\bar{a}) + |a|^2 = r^2.$$

What happens to C under the map $1/z$? More specifically:

(1) If $0 \notin C$, define

$$D = \{1/z \,|\, z \in C\}.$$

Show that D is also a circle, and calculate its center and radius.

(2) If $0 \in C$, then instead define

$$D = \{1/z \,|\, z \in C,\ z \neq 0\}.$$

Describe the set D geometrically. Prove your description is correct.