Topology II Homework 1

Problem 1: Compute the simplicial homology of the Klein bottle using the Δ-complex structure, with two simplices of dimesnion 2.

Problem 2: Show that if G is a finitely generated free abelian group and $H \subset G$ is a subgroup, then there is a basis $g_1, \ldots, g_n$ for G and integers $p_1, \ldots, p_k$ with $k \leq n$ such that each p_i divides p_{i+1}, and such that $p_1 g_1, \ldots, p_k g_k$ is a basis for H. We say that these bases for G and H are *stacked*. Conclude that

$$G/H \cong \mathbb{Z}_{p_1} \oplus \mathbb{Z}_{p_2} \oplus \cdots \oplus \mathbb{Z}_{p_k} \oplus \mathbb{Z}^{n-k}.$$

In particular, every finitely generated abelian group is a direct sub of cyclic groups. (Hint: You may find it helpful to use the fact that subgroups of free abelian groups are themselves free abelian.)

Problem 3: If $\iota : A \to X$ is the inclusion of a retract of X, show that $\iota_* : \mathrm{H}_k(A) \to \mathrm{H}_k(X)$ is a monomorphism onto a direct summand of $\mathrm{H}_k(X)$. If A is a deformation retract of X, show that ι_* is an isomorphism.

Problem 4: Show that it is impossible to retract the n–ball B^n onto its $(n-1)$–sphere boundary $\partial B^n = S^{n-1}$.

Problem 5: In the context of our proof of the zig-zag lemma. Prove that $\mathrm{Ker}(\phi_*) \subset \mathrm{Im}(\partial_*)$ and $\mathrm{Ker}(\psi_*) \subset \mathrm{Im}(\phi_*)$

Problem 6: Let $A : S^n \to S^n$ be the antipodal map. What is $A_* : \mathrm{H}_n(S^n) \to \mathrm{H}_n(S^n)$?