HEEGAARD FLOER HOMOLOGY AND RIBBON HOMOLOGY COBORDISMS

TYE LIDMAN, DAVID SHEA VELA-VICK, AND C.-M. MICHAEL WONG

Abstract. It has recently been shown by several authors that ribbon concordances, or certain variants thereof, induce an injection on knot homology theories. We prove a variant of this for Heegaard Floer homology and homology cobordisms without 3-handles.

1. Introduction

Building on work of Zemke [Zem19], several authors [JMZ19, LZ19, MZ19, Sar19] have studied the behavior of knot homology theories under certain concordances. In the simplest form, if $C\colon K \to K'$ is a ribbon concordance in $S^3 \times I$, then it is shown that the associated cobordism map F_C includes $H(K)$ into $H(K')$ as a summand, where H denotes knot Floer homology or Khovanov homology. The goal of this note is to prove an analogue of this for cobordism maps on Heegaard Floer homology for 3-manifolds. We say that a smooth, connected, oriented cobordism between two connected, oriented, closed 3-manifolds is *ribbon* if it admits a handle decomposition with no 3-handles. (We assume, as per convention, that the handle decomposition has no 0- or 4-handles.) Our main theorem is:

Theorem 1.1. *Let Y and Y' be rational-homology spheres, and let W be a ribbon $\mathbb{Z}/2$-homology cobordism from Y to Y'. Then the cobordism map F°_W includes $\mathrm{HF}^\circ(Y)$ into $\mathrm{HF}^\circ(Y')$ as a summand. Here, $\circ$ denotes any flavor of Heegaard Floer homology.*

Remark 1.2. In fact, we prove a Spin^c-refinement of the above; see Theorem 5.2 and Corollary 5.3.

We provide some examples of such manifolds here. First, if C is a ribbon concordance in $S^3 \times I$, then the double cover of $S^3 \times I$ branched over C is a ribbon $\mathbb{Z}/2$-homology cobordism. Another large family of examples comes from any $\mathbb{Z}/2$-homology cobordism that admits a Stein structure.

Corollary 1.3. *Let $C \subset S^3 \times I$ be a ribbon concordance from $K \subset S^3$ to $K' \subset S^3$. Then the cobordism map $F^\circ_{\Sigma(C)}$ includes $\mathrm{HF}^\circ(\Sigma(K))$ into $\mathrm{HF}^\circ(\Sigma(K'))$ as a summand.* □

Corollary 1.4. *Let Y and Y' be rational-homology spheres, and let W be a $\mathbb{Z}/2$-homology cobordism from Y to Y' with a Stein structure. Then the cobordism map F°_W includes $\mathrm{HF}^\circ(Y)$ into $\mathrm{HF}^\circ(Y')$ as a summand.* □

Remark 1.5. Theorem 1.1 is easily seen to be false without the assumption on the non-existence of 3-handles. For example, consider a contractible 4-manifold W bounded by $\Sigma(2,3,13)$. Then, puncturing W induces a cobordism from $-\Sigma(2,3,13)$ to $-S^3$. The former has $\dim_{\mathbb{F}} \widehat{\mathrm{HF}} = 5$, while the latter has $\dim_{\mathbb{F}} \widehat{\mathrm{HF}} = 1$. On the other hand, for the infinity flavor of Heegaard Floer homology,

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Theorem 1.1 can easily be deduced from [OSz03, Proposition 9.9], and its conclusion is true even in the presence of 3-handles.

Remark 1.6. Theorem 1.1 was motivated by an analogue in instanton Floer homology first observed by Daemi.

Remark 1.7. We expect the involutive Heegaard Floer homology of Y also to be a summand of that of Y'.

Remark 1.8. It is possible that the argument described below can be ported to the framework of sutured Floer homology to reprove the analogous statements for ribbon concordances.

Throughout the paper, unless otherwise specified, Heegaard Floer homologies will have coefficients in $\mathbb{F} = \mathbb{Z}/2$ and singular homology will have coefficients in $\mathbb{Z}$.

2. The double as a surgery

Recall that the *double* $D(W)$ of a cobordism W from Y to Y' is formed by gluing W and $-W$ along Y'. In analogy with the arguments used in ribbon concordance, our strategy to prove Theorem 1.1 will be to prove the cobordism map on Heegaard Floer homology induced by $D(W)$ is the identity map. First, we need a topological description of $D(W)$. Note that a ribbon $\mathbb{Z}/2$-homology cobordism has the same number of 1- and 2-handles.

Proposition 2.1. *Let Y and Y' be connected, oriented, closed 3-manifolds, and let W be a ribbon $\mathbb{Z}/2$-homology cobordism from Y to Y', where the number of 1- and 2-handles is m. Then, $D(W)$ can be described by surgery on $X = (Y \times I) \# m(S^1 \times S^3)$ along m disjoint simple closed curves $\gamma_1, \dots, \gamma_m$. If Y is a rational-homology sphere, then $[\gamma_1] \wedge \cdots \wedge [\gamma_m] = \alpha_1 \wedge \cdots \wedge \alpha_m \in \Lambda^*(H_1(X)/\operatorname{Tors}) \otimes \mathbb{F}$, where $\alpha_i \in H_1(X)$ is homologous to the core of the i^{th} $S^1 \times S^3$ summand.*

See Figure 1 for a schematic diagram when $m = 1$.

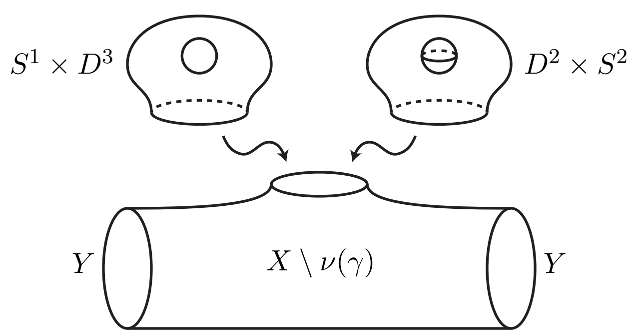


Figure 1. An illustration of Proposition 2.1 in the case of $m = 1$. Here, $\nu(\gamma)$ denotes a neighborhood of γ. Reattaching the $S^1 \times D^3$ would yield $X = (Y \times I) \# (S^1 \times S^3)$, while we may obtain $D(W)$ by switching it for the $D^2 \times S^2$.

Before we prove Proposition 2.1, we first establish an elementary fact.

Lemma 2.2. *Let N be a cobordism from M to M' associated to attaching an n-dimensional k-handle h, where M and M' are $(n-1)$-manifolds. Then the double $D(N)$ can be described by surgery on $M \times I$ along some S^{k-1} which is homologous to the attaching sphere of h.*

Proof. Write $D(N) = (M \times I) \cup h \cup h' \cup (M \times I)$, where h' is the dual handle of h. The cocore of h and the core of h' together form an S^{n-k} with trivial normal bundle, which may be identified with $h \cup h'$. (The case where $n = 4$ and $k = 2$ is described, for example, in [GS99, Example 4.6.3].) Note that h meets the lower $M \times I$, and h' meets the upper $M \times I$, at the same attaching region $S^{k-1} \times D^{n-k} \subset M$, with the same framing. Thus, removing $h \cup h'$ from $D(N)$ would result in $(M \times I) \setminus (S^{k-1} \times D^{n-k} \times (-\epsilon, \epsilon))$. In other words, $D(N)$ may be formed by removing $S^{k-1} \times D^{n-k} \times (-\epsilon, \epsilon) \cong S^{k-1} \times D^{n-k+1}$ from $M \times I$ and replacing it with $h \cup h' \cong D^k \times S^{n-k}$, which is the definition of surgery. □

Proof of Proposition 2.1. First, decompose W into a cobordism W_1 from Y to $\widetilde{Y} = Y \# m(S^1 \times S^2)$, and a cobordism W_2 from $\widetilde{Y}$ to Y', respectively, corresponding to the attachment of 1- and 2-handles. Below, we will compare $D(W) = W_1 \cup W_2 \cup (-W_2) \cup (-W_1)$ with $W_1 \cup (-W_1)$.

Applying Lemma 2.2 to each of the 2-handles in W_2, we see that $W_2 \cup (-W_2)$ can be described by surgery on $\widetilde{Y} \times I$ along some $\gamma_1, \dots, \gamma_m$, where the γ_i's are homologous to the attaching circles of the 2-handles. (Perform isotopies and handleslides first, if necessary, to ensure that the attaching regions of the 2-handles lie in $\widetilde{Y}$ and are disjoint.)

Note that $W_1 \cup (-W_1) \cong W_1 \cup (\widetilde{Y} \times I) \cup (-W_1)$ is diffeomorphic to $X = (Y \times I) \# m(S^1 \times S^3)$. Thus, we see that $D(W) = W_1 \cup W_2 \cup (-W_2) \cup (-W_1)$ can be described by surgery on X along $\gamma_1, \dots, \gamma_m$.

Finally, suppose Y is a rational-homology sphere. Observe that $H_1(X) \cong H_1(Y) \oplus \mathbb{Z}^m$, with $H_1(Y)$ a torsion group. Since W is a $\mathbb{Z}/2$-homology cobordism, the homology classes $[\gamma_i]$'s form a basis for $(H_1(X)/\operatorname{Tors}) \otimes \mathbb{F}$, which is generated by the α_i's; thus, their wedge products are equal. □

3. Cobordism maps in Heegaard Floer theory

Our strategy to prove Theorem 1.1 will be to show that the cobordism map for $D(W)$ is actually just determined by that for $X = (Y \times I) \# k(S^1 \times S^3)$ and the homology classes of the γ_i's, and hence must agree with that of $Y \times I$. We will focus only on $\widehat{\mathrm{HF}}$ in this section; it will be shown later in the proof of Theorem 1.1 that this is sufficient to recover the result for the other flavors. The necessary tool is Proposition 3.2 below, which shows the behavior of the Heegaard Floer cobordism maps under surgery along circles. We expect that this statement is known to experts, as it is already established in Seiberg–Witten theory. See, for example, [KLS19, Corollary 1.4]. For completeness, we provide a proof in this section.

Recall that given a Spin^c-cobordism $(W, \mathfrak{t}) : (Y_1, \mathfrak{s}_1) \to (Y_2, \mathfrak{s}_2)$, Ozsváth and Szabó [OSz06] define cobordism maps

$$F^\circ_{W,\mathfrak{t}} \colon \mathrm{HF}^\circ(Y_1, \mathfrak{s}_1) \otimes \Lambda^*(H_1(W)/\operatorname{Tors}) \to \mathrm{HF}^\circ(Y_2, \mathfrak{t}_2).$$

These maps have the property that

$$F^\circ_{W,\mathfrak{t}}(x \otimes \xi) = F^\circ_{W,\mathfrak{t}}(\xi_1 \cdot x) + \xi_2 \cdot F^\circ_{W,\mathfrak{t}}(x), \tag{3.1}$$

whenever $\xi \in H_1(W)/\operatorname{Tors}$ satisfies $\xi = \iota_1(\xi_1) - \iota_2(\xi_2)$, where $\xi_i \in H_1(Y_i)/\operatorname{Tors}$ and ι_i is induced by inclusion; see [OSz03, p. 186]. We may also sum over all Spin^c-structures on W, and obtain a total map

$$F^\circ_W \colon \mathrm{HF}^\circ(Y_1) \otimes \Lambda^*(H_1(W)/\operatorname{Tors}) \to \mathrm{HF}^\circ(Y),$$

satisfying a property analogous to (3.1). We are now ready to state:

Proposition 3.2. *Let Y_1 and Y_2 be connected, oriented, closed 3-manifolds, and let X be a smooth, connected, oriented cobordism from Y_1 to Y_2. Let $\gamma_1, \dots, \gamma_m \subset \operatorname{Int}(X)$ be loops with disjoint* *neighborhoods* $\nu(\gamma_i) \cong \gamma_i \times D^3$. *Let* Z *be the result of surgery on* X *along* $\gamma_1, \dots, \gamma_m$. *Then for* $x \in \widehat{\mathrm{HF}}(Y_1)$,

$$\widehat{F}_X(x \otimes ([\gamma_1] \wedge \cdots \wedge [\gamma_m])) = \widehat{F}_Z(x). \tag{3.3}$$

Thus, $\widehat{F}_Z$ *depends only on* X *and* $[\gamma_1] \wedge \cdots \wedge [\gamma_m] \in \Lambda^*(H_1(X)/\operatorname{Tors}) \otimes \mathbb{F}$.

There is a Spin^c-refinement of Proposition 3.2; see Proposition 5.4. For the rest of this section, we will mostly postpone the discussion of Spin^c-structures to Section 5.

Before giving the proof, we describe the idea informally. Surgery on γ_i is the result of removing a copy of $S^1 \times D^3$ and replacing it with $D^2 \times S^2$. The cobordism map for $D^2 \times S^2$ agrees with that of $S^1 \times D^3$ if one contracts the latter map by the generator of H_1. Composing with the cobordism map for $X \setminus (\coprod \nu(\gamma_i))$, the result follows. However, to prove this carefully, we must cut and re-glue several different codimension-0 submanifolds, and thus need to use the graph TQFT framework by Zemke [Zem15]. Below, we give a brief review of the necessary elements.

Let Y be a possibly disconnected 3-manifold, and let $\mathbf{p}$ be a set of points in Y with at least one point in each component. Let W be a smooth, connected, oriented cobordism from Y_1 to Y_2, and let Γ be a graph embedded in W with $\partial\Gamma = \mathbf{p}_1 \cup \mathbf{p}_2$. Then, Zemke [Zem15] constructs Heegaard Floer homology groups $\widehat{\mathrm{HF}}(Y_i, \mathbf{p}_i)$ and cobordism maps $\widehat{F}_{W,\Gamma} \colon \widehat{\mathrm{HF}}(Y_1, \mathbf{p}_1) \to \widehat{\mathrm{HF}}(Y_2, \mathbf{p}_2)$.

Theorem 3.4 (Zemke [Zem15])**.** *The cobordism maps* $\widehat{F}_{W,\Gamma}$ *satisfy the following:*

(1) *Under disjoint union, we have that* $\widehat{\mathrm{HF}}(Y_1 \sqcup Y_2, \mathbf{p}_1 \sqcup \mathbf{p}_2) = \widehat{\mathrm{HF}}(Y_1, \mathbf{p}_1) \otimes \widehat{\mathrm{HF}}(Y_2, \mathbf{p}_2)$, *and* $\widehat{F}_{(W_1,\Gamma_1)\sqcup(W_2,\Gamma_2)} = \widehat{F}_{(W_1,\Gamma_1)} \otimes \widehat{F}_{(W_2,\Gamma_2)}$*; see* [Zem15, p. 17].

(2) *Given* $(W, \Gamma) \colon (Y_1, \mathbf{p}_1) \to (Y_2, \mathbf{p}_2)$ *and* $(W', \Gamma') : (Y_2, \mathbf{p}_2) \to (Y_3, \mathbf{p}_3)$, *then* $\widehat{F}_{W',\Gamma'} \circ \widehat{F}_{W,\Gamma} = \widehat{F}_{W\cup W',\Gamma\cup\Gamma'}$*; see* [Zem15, Theorem A (2)].

(3) $\widehat{F}_{W,\Gamma}$ *admits a decomposition by* Spin^c*-structures in the usual way. In particular,* $\widehat{F}_{W,\Gamma} = \sum_{\mathfrak{t}\in\mathrm{Spin}^c(W)} \widehat{F}_{W,\Gamma,\mathfrak{t}}$, *and*

$$\widehat{F}_{W',\Gamma',\mathfrak{t}_{W'}} \circ \widehat{F}_{W,\Gamma,\mathfrak{t}_W} = \sum_{\substack{\mathfrak{t}\in\mathrm{Spin}^c(W\cup W') \\ \mathfrak{t}|_W=\mathfrak{t}_W,\, \mathfrak{t}|_{W'}=\mathfrak{t}_{W'}}} \widehat{F}_{W\cup W',\Gamma\cup\Gamma',\mathfrak{t}};$$

see [Zem15, Theorem B (b) and (c)]. *(We take the convention that this equation remains valid when* $\mathfrak{t}_W|_{Y_2} \neq \mathfrak{t}_{W'}|_{Y_2}$, *in which case both sides of the equation are identically zero.)*

(4) *If* (W, Γ) *and* (W, Γ') *are cobordisms from* $(Y_1, \mathbf{p}_1)$ *to* $(Y_2, \mathbf{p}_2)$ *with diffeomorphic exteriors, then* $\widehat{F}_{W,\Gamma} = \widehat{F}_{W,\Gamma'}$*; see* [Zem15, Remark 3.1]. *Also, if* λ *is an arc from the boundary of some* $B^4 \subset W$ *to* Γ, *then* $\widehat{F}_{W,\Gamma}(x) = \widehat{F}_{W\setminus B^4,\Gamma\cup\lambda}(x \otimes y)$, *where* y *is the generator of* $\widehat{\mathrm{HF}}(\partial B^4)$*; see* [Zem15, Lemma 6.1]. *See* [Zem15, Figure 16] *for an illustration of both of these principles.*

(5) *Suppose that* Y_1 *and* Y_2 *are connected,* $\mathbf{p}_1$ *and* $\mathbf{p}_2$ *each consist of a single point, and* Γ *is a path. Then* $\widehat{F}_W(x) = \widehat{F}_{W,\Gamma}(x)$, *where* $\widehat{F}_W$ *is the original Ozsváth–Szabó cobordism map; see* [Zem15, Theorem A (1)]. *(Implicitly, the Ozsváth–Szabó cobordism map requires a choice of basepoints and a choice of path, but the injectivity statement in Theorem 1.1 is independent of both choices.)*

(6) *Suppose again that* Y_1 *and* Y_2 *are connected,* $\mathbf{p}_1$ *and* $\mathbf{p}_2$ *each consist of a single point, and* Γ *is a path. Let* γ *be a simple closed loop in* $\operatorname{Int}(W)$ *that intersects* Γ *at a single point. Then* $\widehat{F}_W(x \otimes [\gamma]) = \widehat{F}_{W,\Gamma\cup\gamma}(x)$, *where the left-hand side is the Ozsváth–Szabó cobordism map defined above; see* [Zem15, Theorem C].

(7) *Let* Y *be connected and let* $\mathbf{p}$ *consist of a single point. Consider* $\Gamma = \mathbf{p} \times I \subset Y \times I$. *Choose a simple closed loop* γ *in* Y *based at* $\mathbf{p}$ *and let* Γ_γ *be the graph obtained by appending* $\gamma \times \{1/2\}$

to Γ. *Denote the cobordism map* $\widehat{F}_{Y\times I,\Gamma_\gamma}$ *by* $\mathcal{F}(\gamma)$. *Then,* $\mathcal{F}(\gamma)$ *depends only on* $[\gamma] \in H_1(Y)$ *(*[Zem15, Corollary 7.4]*). Furthermore,* $\mathcal{F}(\gamma * \gamma') = \mathcal{F}(\gamma) + \mathcal{F}(\gamma')$ *(*[Zem15, Lemma 7.3]*) and* $\mathcal{F}(\gamma) \circ \mathcal{F}(\gamma) = 0$ *(*[Zem15, Lemma 7.2]*). Here,* $\gamma * \gamma'$ *is a simple closed loop in the based homotopy class of the concatenation.*

We now need a slight generalization of Theorem 3.4 (6), i.e. [Zem15, Theorem C], which will allow us to analyze the effect, on the cobordism map, of appending multiple loops to a path. We begin with the identity cobordism.

Lemma 3.5. *Let* Y *be connected and let* $\mathbf{p}$ *consist of a single point. Let* Γ *be a graph obtained by taking* $\mathbf{p} \times I \subset Y \times I$ *and appending to it* m *disjoint simple closed curves* $\gamma_1, \dots, \gamma_m$*, which each intersect* $\mathbf{p} \times I$ *only at a single point. Then*

$$\widehat{F}_{Y\times I}(x \otimes ([\gamma_1] \wedge \cdots \wedge [\gamma_m])) = \widehat{F}_{Y\times I,\Gamma}(x),$$

where the left-hand side is the Ozsváth–Szabó cobordism map.

Proof. This is implicit in the work of Zemke [Zem15], but we give the proof for completeness. By a homotopy, and hence isotopy, in $Y \times I$, we may arrange that $\gamma_i \subset Y \times \{i/(m+1)\}$. Therefore, using Theorem 3.4 (2), we can write $\widehat{F}_{Y\times I,\Gamma}$ as a composition of the maps $\mathcal{F}(\gamma_i)$. Viewing $\mathcal{F}$ as a function from $H_1(Y)$ to $\mathrm{End}_{\mathbb{F}}(\widehat{\mathrm{HF}}(Y))$, Theorem 3.4 (7) implies that this descends to the exterior algebra. □

We move on to more general cobordisms.

Lemma 3.6. *Let* Y_1 *and* Y_2 *be connected, let* $\mathbf{p}_1$ *and* $\mathbf{p}_2$ *each consist of a single point, and let* W *be a connected cobordism from* Y_1 *to* Y_2*. Let* Γ *be a graph obtained by taking a path* α *from* $\mathbf{p}_1$ *to* $\mathbf{p}_2$ *and appending to it* m *disjoint simple closed loops* $\gamma_1, \dots, \gamma_m$*, which each intersect* α *only at a single point. Then*

$$\widehat{F}_{W}(x \otimes ([\gamma_1] \wedge \cdots \wedge [\gamma_m])) = \widehat{F}_{W,\Gamma}(x),$$

where the left-hand side is the Ozsváth–Szabó cobordism map.

Proof. This follows the proof of [Zem15, Theorem C]. We may decompose (W, Γ) as a composition of three cobordisms: (W_1, Γ_1), where W_1 consists only of 1-handles and Γ_1 is a path; $(\partial W_1 \times I, \Gamma_*)$, where Γ_* consists of a graph in $Y \times I$ as in the statement of Lemma 3.5; and (W_2, Γ_2), where W_2 consists of 2- and 3-handles, and Γ_2 is again a path. The result now follows from Lemma 3.5 together with Theorem 3.4 (2). □

With this generalization, we may now complete the proof of Proposition 3.2.

Proof of Proposition 3.2. Let $\nu(\gamma_i) \cong \gamma_i \times D^3$ be a neighborhood of γ_i, and let $P = X \setminus (\coprod_i \nu(\gamma_i))$. Let $X' = X \setminus (B^4_1 \sqcup \cdots \sqcup B^4_m)$, where $B^4_i \subset \mathrm{Int}(\nu(\gamma_i))$. We construct a properly embedded graph $\Gamma_{X'}$ in X' as follows; see Figure 2.

We begin with the vertex set. Choose m points $p_1, \dots, p_m$ in the interior of P, and points p_0 and p_{m+1} in Y_1 and Y_2 respectively. Choose m points $q_1, \dots q_m$ with $q_i \in \partial\nu(\gamma_i)$, which are copies of $S^1 \times S^2$. Choose m points $r_1, \dots, r_m$ with $r_i \in \gamma_i$. Finally, let s_i be a point in $S^3_i = \partial B^4_i$ for each i.

Now we define the edge sets. Choose any collection of embedded arcs $\alpha_0, \dots, \alpha_m$ with $\alpha_i \subset P$ connecting p_i and p_{i+1}. Let $\beta_i \subset P$ be an arc from p_i to q_i. Connect q_i and r_i by arcs δ_i, and r_i and s_i by arcs ϵ_i, in $\nu(\gamma_i) \setminus B^4_i$. We may choose the edges above in such a way that their interiors are mutually disjoint, avoid the γ_i, and are contained in the interior of X'. Then, the edge set of

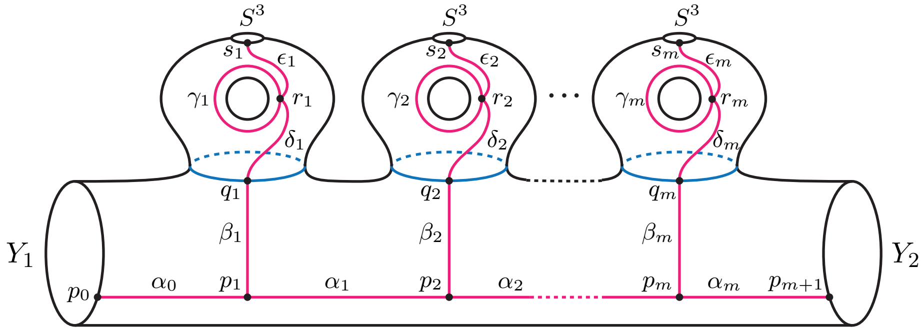


FIGURE 2. The embedded graph $\Gamma_{X'}$ in X'.

$\Gamma_{X'}$ consists of the edges α_i, β_i, γ_i, δ_i, and ϵ_i. In accordance with Theorem 3.4 (1), we view the cobordism map for $(X', \Gamma_{X'})$ as a map

$$\widehat{F}_{X',\Gamma_{X'}}\colon \widehat{\mathrm{HF}}(Y_1) \otimes \left(\bigotimes_{i=1}^{m} \widehat{\mathrm{HF}}(S^3_i) \right) \to \widehat{\mathrm{HF}}(Y_2).$$

It follows from Lemma 3.6 as well as Theorem 3.4 (1) and (4) that

$$\widehat{F}_X(x \otimes ([\gamma_1] \wedge \cdots \wedge [\gamma_m])) = \widehat{F}_{X',\Gamma_{X'}}(x \otimes y_1 \otimes \cdots \otimes y_m),$$

where y_i is the generator of $\widehat{\mathrm{HF}}(S^3_i)$. (We can first contract the homology elements, and then contract the arcs $\beta_i \cup \delta_i \cup \epsilon_i$.) Let Γ_P be the intersection of $\Gamma_{X'}$ with P, which can alternatively be obtained by excising the γ_i, δ_i, and ϵ_i arcs.

Note that $Z = P \cup (\coprod_i (D^2 \times S^2)_i)$. Here, we suppress the choice of gluing from the notation. Similarly, we let $Z' = Z \setminus (B^4_1 \sqcup \cdots \sqcup B^4_m)$ where $B^4_i \subset (D^2 \times S^2)_i$; then $Z' = P \cup (\coprod_i R_i)$, where each R_i is a punctured $D^2 \times S^2$. Let ζ_i be an arc in R_i that connects q_i and s_i; then we define Γ_{R_i} in R_i to be ζ_i, and define $\Gamma_{Z'}$ in Z' as the union of the arcs $\alpha_i, \beta_i, \zeta_i$. See Figure 3 for an illustration of $(Z', \Gamma_{Z'})$.

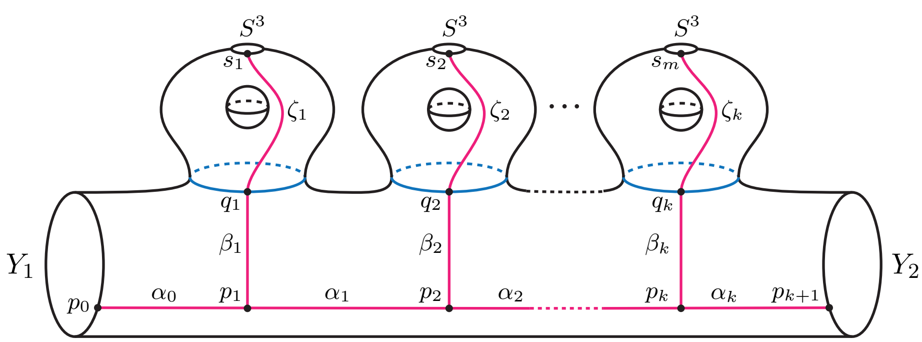


FIGURE 3. The embedded graph $\Gamma_{Z'}$ in Z'.

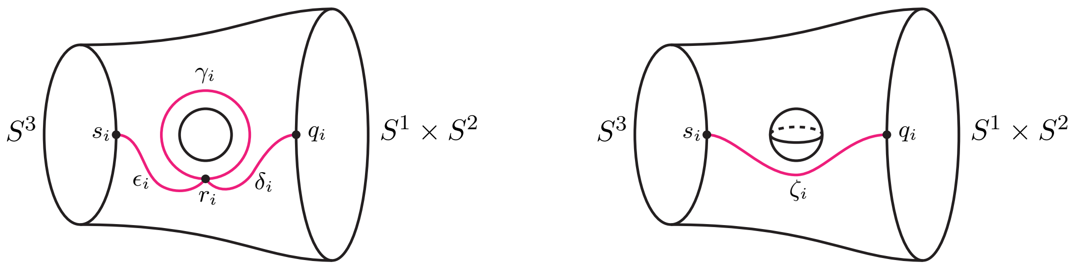
Viewing the cobordism map for $(Z', \Gamma_{Z'})$ as a map $\widehat{F}_{Z',\Gamma_{Z'}}\colon \widehat{\mathrm{HF}}(Y_1) \otimes (\bigotimes_i \widehat{\mathrm{HF}}(S^3_i)) \to \widehat{\mathrm{HF}}(Y_2)$, we have

$$\widehat{F}_Z(x) = \widehat{F}_{Z',\Gamma_{Z'}}(x \otimes y_1 \otimes \cdots \otimes y_m),$$

again by Theorem 3.4 (4). Thus, (3.3) will follow if we can show

$$\widehat{F}_{X',\Gamma_{X'}}(x \otimes y_1 \otimes \cdots \otimes y_m) = \widehat{F}_{Z',\Gamma_{Z'}}(x \otimes y_1 \otimes \cdots \otimes y_m).$$

To do so, let $Q_i = \nu(\gamma_i) \setminus B_i^4$, and let Γ_{Q_i} be the intersection of $\Gamma_{X'}$ with Q_i. Both (Q_i, Γ_{Q_i}) and (R_i, Γ_{R_i}) are cobordisms from (S^3, s_i) to $(S^1 \times S^2, q_i)$; see Figure 4.



(A) The cobordism (Q_i, Γ_{Q_i}). (B) The cobordism (R_i, Γ_{R_i}).

FIGURE 4. The cobordisms (Q_i, Γ_{Q_i}) and (R_i, Γ_{R_i}).

Viewing (P, Γ_P) as a cobordism from $(Y_1, p_0) \sqcup (\coprod_i (S^1 \times S^2)_i, q_i) \to (Y_2, p_{k+1})$, by Theorem 3.4 (1) and (2), we have that

$$\widehat{F}_{X',\Gamma_{X'}} = \widehat{F}_{P,\Gamma_P} \circ \left(\mathbb{I}_{\widehat{\mathrm{HF}}(Y_1)} \otimes \widehat{F}_{Q_1,\Gamma_{Q_1}} \otimes \cdots \otimes \widehat{F}_{Q_m,\Gamma_{Q_m}}\right)$$

and

$$\widehat{F}_{Z',\Gamma_{Z'}} = \widehat{F}_{P,\Gamma_P} \circ \left(\mathbb{I}_{\widehat{\mathrm{HF}}(Y_1)} \otimes \widehat{F}_{R_1,\Gamma_{R_1}} \otimes \cdots \otimes \widehat{F}_{R_m,\Gamma_{R_m}}\right).$$

Thus, we need only to show that $\widehat{F}_{Q_i,\Gamma_{Q_i}} = \widehat{F}_{R_i,\Gamma_{R_i}}$ for each i. On the one hand, Theorem 3.4 (6) together with (3.1) imply that

$$\widehat{F}_{Q_i,\Gamma_{Q_i}}(y_i) = \widehat{F}_{Q_i}(y_i \otimes [\gamma_i]) = [\gamma_i] \cdot \widehat{F}_{Q_i}(y_i).$$

Since Q_i is simply a 1-handle attachment to S^3, its cobordism map, by Ozsváth and Szabó's definition, sends y_i to the topmost generator of $\widehat{\mathrm{HF}}(S^1 \times S^2)$, and the action by $[\gamma_i]$ sends this to the bottommost generator. On the other hand, Theorem 3.4 (5) implies that

$$\widehat{F}_{R_i,\Gamma_{R_i}}(y_i) = \widehat{F}_{R_i}(y_i).$$

Since R_i is simply a 0-framed 2-handle attachment along the unknot in S^3, its cobordism map sends y_i to the bottommost generator of $\widehat{\mathrm{HF}}(S^1 \times S^2)$. Consequently, $\widehat{F}_{Q_i,\Gamma_{Q_i}}(y_i) = \widehat{F}_{R_i,\Gamma_{R_i}}(y_i)$ as desired. $\square$

4. PROOF OF THEOREM 1.1

Proof of Theorem 1.1. We consider the hat flavor first. Consider the double $D(W)$ of W. Then, by Proposition 2.1, $D(P)$ is described by surgery on $X = (Y \times I) \# m(S^1 \times S^3)$ along m circles $\gamma_1, \ldots, \gamma_m$, where $[\gamma_1] \wedge \cdots \wedge [\gamma_m] = \alpha_1 \wedge \cdots \wedge \alpha_m \in \Lambda^*(H_1(X)/\operatorname{Tors}) \otimes \mathbb{F}$, and α_i is homologous to the core of the i^{th} $S^1 \times S^3$ summand. Note that the same description is true of $Y \times I$; in this case, the surgery is performed along the core circles γ_i' of the $(S^1 \times S^3)$'s themselves. Clearly,

$$[\gamma_1] \wedge \cdots \wedge [\gamma_m] = [\gamma_1'] \wedge \cdots \wedge [\gamma_m'] \in \Lambda^*(H_1(X)/\operatorname{Tors}) \otimes \mathbb{F}.$$

Applying Proposition 3.2 with $Z = D(W)$, we have that

$$\widehat{F}_X(x \otimes ([\gamma_1] \wedge \cdots \wedge [\gamma_m])) = \widehat{F}_{D(W)}(x) = \widehat{F}_{-W} \circ \widehat{F}_W(x). \tag{4.1}$$

Now consider $Y \times I$ as surgery on X along the cores γ_i'. Applying Proposition 3.2 again, this time with $Z = Y \times I$, we have

$$\widehat{F}_X(x \otimes ([\gamma_1'] \wedge \cdots \wedge [\gamma_m'])) = \widehat{F}_{Y\times I}(x),$$

which is the identity on $\widehat{\mathrm{HF}}(Y)$. This computes the left-hand side of (4.1), implying that

$$\widehat{F}_{-W} \circ \widehat{F}_W = \mathbb{I}_{\widehat{\mathrm{HF}}(Y)},$$

and we have the desired result for $\widehat{\mathrm{HF}}$. To obtain the analogous result for the other flavors of Heegaard Floer homology, we use that the long exact sequences relating the various flavors are natural with respect to cobordism maps. It is straightforward to see that only the identity map on HF^+ can induce the identity map on $\widehat{\mathrm{HF}}$, and similarly for HF^-. Finally, only the identity map on HF^∞ can induce the identity on both HF^+ and HF^-. □

5. Spin^c-structures

We now provide a Spin^c-refinement of Theorem 1.1. We begin with an observation.

Lemma 5.1. *Let Y and Y' be rational-homology spheres, and let W be a ribbon $\mathbb{Z}/2$-homology cobordism from Y to Y'. If a* Spin^c*-structure $\mathfrak{s}'$ on Y' can be extended to a* Spin^c*-structure $\mathfrak{t}$ on W, then the extension is unique; moreover, in this case, $\mathfrak{t}$ can be extended to a unique* Spin^c*-structure $D(\mathfrak{t})$ on $D(W)$.*

Proof. For the first statement, consider

$$0 = H^1(Y') \to H^2(W, Y') \to H^2(W) \to H^2(Y')$$

from the long exact sequence of the pair (W, Y'). On the one hand, building W from Y, the 2-handles are attached along circles whose homology classes are linearly independent, and so $H_2(W) = 0$, and $H^2(W)$ is finite. On the other hand, viewing W upside down, it is built from Y' by adding 2- and 3-handles only, and so $H^2(W, Y')$ is free. This shows that $H^2(W, Y') = 0$; thus, the map $H^2(W) \to H^2(Y')$ induced by inclusion is injective, proving that any extension $\mathfrak{t}$ of $\mathfrak{s}'$ is unique.

For the second statement, note first that $\mathfrak{t}$ on W and $\mathfrak{t}$ on $-W$ coincide on the intersection $W \cap -W = Y'$, and so there is an extension of $\mathfrak{t}$ to $D(W)$. Considering

$$0 = H^1(Y') \to H^2(D(W)) \to H^2(W) \oplus H^2(-W)$$

from the Mayer–Vietoris exact sequence, we see that the second map is again injective, showing that this extension is again unique. □

Lemma 5.1 implies that for distinct $\mathfrak{t}_1, \mathfrak{t}_2 \in \mathrm{Spin}^c(W)$, their restrictions $\mathfrak{t}_1|_{Y'}, \mathfrak{t}_2|_{Y'} \in \mathrm{Spin}^c(Y')$ are distinct. We are now ready to state the following refinement of Theorem 1.1:

Theorem 5.2. *Let Y and Y' be rational-homology spheres, and let W be a ribbon $\mathbb{Z}/2$-homology cobordism from Y to Y'. Fix a* Spin^c*-structure $\mathfrak{s}$ on Y. Then the sum of cobordism maps*

$$\left(\sum_{\substack{\mathfrak{t}\in\mathrm{Spin}^c(W)\\ \mathfrak{t}|_Y=\mathfrak{s}}} F^\circ_{W,\mathfrak{t}}\right) : \mathrm{HF}^\circ(Y,\mathfrak{s}) \to \bigoplus_{\substack{\mathfrak{t}\in\mathrm{Spin}^c(W)\\ \mathfrak{t}|_Y=\mathfrak{s}}} \mathrm{HF}^\circ(Y', \mathfrak{t}|_{Y'})$$

includes $\mathrm{HF}^\circ(Y,\mathfrak{s})$ *into the codomain as a summand.*

With the extra restriction that W is a $\mathbb{Z}$-homology cobordism, a Spin^c structure $\mathfrak{s}$ on Y determines a unique $\mathfrak{t}$ on W, and hence a unique $\mathfrak{s}'$ on Y'. We have:

Corollary 5.3. *Let Y and Y' be rational-homology spheres, and let W be a ribbon $\mathbb{Z}$-homology cobordism from Y to Y'. Fix a Spin^c-structure $\mathfrak{s}$ on Y, and let $\mathfrak{t}$ and $\mathfrak{s}'$ be the corresponding Spin^c structures on W and Y' respectively. Then the cobordism map*

$$F^\circ_{W,\mathfrak{t}} \colon \mathrm{HF}^\circ(Y,\mathfrak{s}) \to \mathrm{HF}^\circ(Y',\mathfrak{s}')$$

includes $\mathrm{HF}^\circ(Y,\mathfrak{s})$ into $\mathrm{HF}^\circ(Y',\mathfrak{s}')$ as a summand. □

The main ingredient to prove Theorem 5.2 is the following analogue of Proposition 3.2.

Proposition 5.4. *Let Y_1 and Y_2 be connected, oriented, closed 3-manifolds, and let X be a smooth, connected, oriented cobordism from Y_1 to Y_2. Let $\gamma_1,\dots,\gamma_m \subset \mathrm{Int}(X)$ be loops with disjoint neighborhoods $\nu(\gamma_i) \cong \gamma_i \times D^3$. Let Z be the result of surgery on X along $\gamma_1,\dots,\gamma_m$, and let $P = X \setminus (\coprod_i \nu(\gamma_i))$. Fix a Spin^c-structure $\mathfrak{t}_P$ on P. Then for $x \in \widehat{\mathrm{HF}}(Y_1, \mathfrak{t}_P|_{Y_1})$,*

$$\sum_{\substack{\mathfrak{t}_X\in\mathrm{Spin}^c(X)\\ \mathfrak{t}_X|_P=\mathfrak{t}_P}} \widehat{F}_{X,\mathfrak{t}_X}(x\otimes([\gamma_1]\wedge\cdots\wedge[\gamma_m])) = \sum_{\substack{\mathfrak{t}_Z\in\mathrm{Spin}^c(Z)\\ \mathfrak{t}_Z|_P=\mathfrak{t}_P}} \widehat{F}_{Z,\mathfrak{t}_Z}(x). \tag{5.5}$$

Proof. We analyze the Spin^c-structures in the proof of Proposition 3.2. Working backwards, note that our argument implies that

$$\sum_{\mathfrak{t}_{Q_i}\in\mathrm{Spin}^c(Q_i)} \widehat{F}_{Q_i,\Gamma_{Q_i},\mathfrak{t}_{Q_i}} = \sum_{\mathfrak{t}_{R_i}\in\mathrm{Spin}^c(R_i)} \widehat{F}_{R_i,\Gamma_{R_i},\mathfrak{t}_{R_i}}.$$

(The astute reader may have noticed that in fact $\mathrm{Spin}^c(Q_i)$ consists of a unique element $\mathfrak{t}_{Q_i}$, all terms on the right-hand side vanish except for the unique self-conjugate $\mathfrak{t}_{R_i}$, and $\mathfrak{t}_{Q_i}|_{S^1\times S^2} = \mathfrak{t}_{R_i}|_{S^1\times S^2}$.) Composing these maps with $\widehat{F}_{P,\Gamma_P,\mathfrak{t}_P}$ and using Theorem 3.4 (3), we see that the rest of the argument in the proof of Proposition 3.2 implies (5.5). □

Note the following consequence of Proposition 5.4.

Proposition 5.6. *Let Y_1, Y_2, X, and Z be as in Proposition 5.4. Fix Spin^c-structures $\mathfrak{s}_i$ on Y_i. Then for $x \in \widehat{\mathrm{HF}}(Y_1,\mathfrak{s}_1)$,*

$$\sum_{\substack{\mathfrak{t}_X\in\mathrm{Spin}^c(X)\\ \mathfrak{t}_X|_{Y_i}=\mathfrak{s}_i}} \widehat{F}_{X,\mathfrak{t}_X}(x\otimes([\gamma_1]\wedge\cdots\wedge[\gamma_m])) = \sum_{\substack{\mathfrak{t}_Z\in\mathrm{Spin}^c(Z)\\ \mathfrak{t}_Z|_{Y_i}=\mathfrak{s}_i}} \widehat{F}_{Z,\mathfrak{t}_Z}(x). \tag{5.7}$$

Proof. The result is obtained by summing both sides in (5.5) over all $\mathfrak{t}_P$ that restrict to $\mathfrak{s}_i$ on Y_i. □

Proof of Theorem 5.2. Again, we analyze the Spin^c-structures in the proof of Theorem 1.1. Fixing a Spin^c-structure $\mathfrak{s}$ on Y, note that $X = (Y\times I)\,\#\,m(S^1\times S^3)$ has a unique Spin^c-structure $\mathfrak{t}_X$ that restricts to $\mathfrak{s}$, since the map $H^2(X)\to H^2(Y)$ induced by inclusion is an isomorphism. Applying Proposition 5.6 with $Z = D(W)$, we have that

$$\widehat{F}_{X,\mathfrak{t}_X}(x\otimes([\gamma_1]\wedge\cdots\wedge[\gamma_m])) = \sum_{\substack{\mathfrak{t}\in\mathrm{Spin}^c(D(W))\\ \mathfrak{t}|_{Y\times\{0\}}=\mathfrak{s},\, \mathfrak{t}|_{Y\times\{1\}}=\mathfrak{s}}} \widehat{F}_{D(W),\mathfrak{t}}(x). \tag{5.8}$$

Next, considering $Y\times I$ as surgery on X along the cores γ_i', we apply Proposition 5.6 again, this time with $Z = Y\times I$, to obtain

$$\widehat{F}_{X,\mathfrak{t}_X}(x\otimes([\gamma_1']\wedge\cdots\wedge[\gamma_m'])) = \widehat{F}_{Y\times I,\mathfrak{s}_*}(x),$$

where $\mathfrak{s}_*$ is the unique Spin^c-structure on $Y \times I$ that restricts to $\mathfrak{s}$. Again, this is the identity on $\widehat{\mathrm{HF}}(Y, \mathfrak{s})$. Turning to the right-hand side of (5.8), Lemma 5.1 implies that there is a one-to-one correspondence between Spin$^c(W)$ and Spin$^c(D(W))$, given by the map $\mathfrak{t} \mapsto D(\mathfrak{t})$. Thus, we see

$$\left(\sum_{\substack{\mathfrak{t} \in \mathrm{Spin}^c(W) \\ \mathfrak{t}|_Y = \mathfrak{s}}} \widehat{F}_{-W,\mathfrak{t}} \right) \circ \left(\sum_{\substack{\mathfrak{t} \in \mathrm{Spin}^c(W) \\ \mathfrak{t}|_Y = \mathfrak{s}}} \widehat{F}_{W,\mathfrak{t}} \right) = \sum_{\substack{\mathfrak{t} \in \mathrm{Spin}^c(W) \\ \mathfrak{t}|_Y = \mathfrak{s}}} \widehat{F}_{D(W),D(\mathfrak{t})} = \mathbb{I}_{\widehat{\mathrm{HF}}(Y,\mathfrak{s})},$$

giving us the result for $\widehat{\mathrm{HF}}$. A long-exact-sequence argument completes the proof for other flavors. ☐

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