Topology I Homework 7/Final

Problem 1:(Pledged) Let X denote the surface of genus two with a single boundary component. let A denote the boundary of X. Compute the relative homology groups $\mathrm{H}_p(X, A)$.

Problem 2:(Pledged) Show that $S^1 \times S^1$ and $S^1 \vee S^1 \vee S^2$ have isomorphic homology groups, but that their universal covering spaces do not.

Problem 3:(Pledged) Group the following spaces into homotopy equivalence classes. Justify your answers.

(a) Möbius band
(b) The torus
(c) $B^2 \times S^1$
(d) The torus minus one point
(e) The torus minus two points
(f) The Klein bottle minus one point
(g) $\mathbb{R}^3$ minus the z-axis
(h) $\mathbb{R}^3$ minus the unit circle in the xy-plane, $\{x^2 + y^2 = 0,\, z = 0\}$
(i) The intersection of (g) and (h)
(j) S^3 minus two linked circles
(k) S^3 minus two unlinked circles

Problem 4: Compute the homology of the CW complex obtained from the cube $I \times I \times I$ by identifying opposite faces after a 1/4 twist.

Problem 5: A map $f : S^n \to S^n$ satisfying $f(x) = f(-x)$ for all $x \in S^n$ is called an *even map*. Show that an even map $S^n \to S^n$ must have even degree, and that the degree must in fact be zero when n is even. When n is odd, show that there exist even maps of any given even degree. [Hint: First show that if f is even, then it necessarily factors as a composition $S^n \to \mathbb{R}P^n \to S^n$.]

Problem 6: Let $\mathcal{C} = (C_i, \partial)$ be a chain complex over $\mathbb{R}$ with only finitely many $C_i \neq 0$. Show that the following two methods for computing Euler characteristic yield the same answer:

$$\chi(\mathcal{C}) = \sum_i (-1)^i \mathrm{rk} C_i$$

and

$$\chi(\mathcal{C}) = \sum_i (-1)^i \mathrm{rk} H_i(\mathcal{C}).$$

The same is true with $\mathbb{Z}$ coefficients, but requires a tiny bit more thought.