

Topology I Homework 5

Problem 1: Verify explicitly that “boundary squared equals zero”. That is, prove that $\partial_p \circ \partial_{p+1} = 0$ for the boundary map in simplicial homology.

Problem 2: Compute the simplicial homology of the Klein bottle using the Δ -complex structure, with two simplices of dimension 2.

Problem 3: Show that if G is a finitely generated free abelian group and $H \subset G$ is a subgroup, then there is a basis g_1, \dots, g_n for G and integers p_1, \dots, p_k with $k \leq n$ such that each p_i divides p_{i+1} , and such that p_1g_1, \dots, p_kg_k is a basis for H . We say that these bases for G and H are *stacked*. Conclude that

$$G/H \cong \mathbb{Z}_{p_1} \oplus \mathbb{Z}_{p_2} \oplus \dots \oplus \mathbb{Z}_{p_k} \oplus \mathbb{Z}^{n-k}.$$

In particular, every finitely generated abelian group is a direct sum of cyclic groups. (Hint: You may find it helpful to use the fact that subgroups of free abelian groups are themselves free abelian.)

Problem 4: If $\iota : A \rightarrow X$ is the inclusion of a retract of X , show that $\iota_* : H_k(A) \rightarrow H_k(X)$ is a monomorphism onto a direct summand of $H_k(X)$. If A is a deformation retract of X , show that ι_* is an isomorphism.

Problem 5: Show that it is impossible to retract the n -ball B^n onto its $(n - 1)$ -sphere boundary $\partial B^n = S^{n-1}$.

Problem 6: Compute the simplicial homology of the Klein bottle using the Δ -complex structure, with two simplices of dimension 2, discussed in class.