Topology I Homework 4

Problem 0: Please read Chapter 1 *The Fundamental Group* of Hatcher's book (pages 21 through 87). Pay especial attention to the examples discussed in this chapter.

Problem 1: Suppose that Σ and Σ' are closed, orientable surfaces, and that $\chi(\Sigma) = n \cdot \chi(\Sigma')$, with $n \in \mathbb{Z}_{>0}$. Show that there exists an n–sheeted covering map $p : \Sigma \to \Sigma'$. How must this result be modified if we want to include non-orientable surfaces?

Problem 2: (Pledged) Let $p : E \to B$ be a covering map. Assume that both B and E are path connected. Show that this covering is normal if and only if for each loop in B, either all its lifts are loops or none of them are loops.

Problem 3: Let B be a path connected, locally path connected and semi-locally simply connected.

- (a) Show that the set of equivalence classes of all n–sheeted covering spaces of B is in one-to-one correspondence with the set of equivalence classes of all representations of $\pi_1(B, b_0)$ as a permutation group on n symbols.
- (b) Show that the covering space is path-connected if and only if the representation is transitive.

Problem 4: (Pledged) Let Σ_g denote the closed surface of genus g. Show that $\Sigma_n \cong \Sigma_m$ if and only if $n = m$.

Problem 5: (Pledged) Let $p : E \to B$ be a covering map. Assume that both B and E are path connected. Show that this cover is normal if and only if the group of covering transformations acts transitively on the set of points in E which lie over any single point in B.

Problem 6: Find all connected covering spaces of $\mathbb{R}P^2 \vee \mathbb{R}P^2$.

Problem 7: For a path connected, locally path connected, and semi-locally simply connected space X, call a covering space $p : \widetilde{X} \to X$ *abelian* if it is normal and has abelian deck transformation group. Show that X has an abelian covering space that is a covering space of every other abelian covering space of X, and that such a "universal" abelian covering space is unique up to isomorphism. Describe this covering space explicitly for $X = S^1 \vee S^1$ and $X = S^1 \vee S^1 \vee S^1$.