Multivariable Analysis Homeowork 4

Problem 1: Let $\mathbb{H}$ denote the space of quaternions. Define the conjugate of a quaternion $p = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$ to be $\bar{p} = x_0 - x_1\mathbf{i} - x_2\mathbf{j} - x_3\mathbf{k}$. Show that $p\bar{p} = |p|^2$ and, in turn, that if p is nonzero $p^{-1} = \bar{p}/|p|^2$

Problem 2: Let $\mathbb{H}$ denote the space of quaternions. Show that $|pq| = |p||q|$ and use it to prove show that:

(a) S^3 is a Lie group.
(b) The map $q \mapsto [v \mapsto qvq^{-1}]$, where $q \in \mathbb{H}$, $|q| = 1$, and $v \in \mathrm{Im}(\mathbb{H})$ is a Lie group homomorphism from S^3 to $\mathrm{SO}(3)$.
(c) The Lie group $\mathrm{SO}(3)$ is diffeomorphic to $\mathbb{R}P^3$.

Problem 3: Show that the map $f : S^3 \times S^3 \to \mathrm{SO}(4)$ defined by

$$f(q, r) = [v \in \mathbb{H} \mapsto qvr^{-1} \in \mathbb{H}]$$

is a Lie group homomorphism which is onto with kernel $\{\pm(1, 1)\}$. Conclude that $\mathrm{SO}(4)$ is diffeomorphic to $S^3 \times S^3/[(x, y) \sim (-x, -y)]$.

Problem 4: Show that $O(n)$ is a manifold. (Hint: Consider the map $T : \mathcal{M}(n \times n) \to Sym(n)$ which takes an $n \times n$ matrix A to the symmetric matrix AA^T. Under this map, $O(n)$ can be viewed as the level set $T^{-1}(Id)$.)