Multivariable Analysis Final

Problem 1:(Pledged) Show that the vectors $v_1, \ldots, v_k \in V$ are linearly independent if and only if $v_1 \wedge \cdots \wedge v_k \neq 0$ in $\bigwedge^k(V)$.

Problem 2:(Pledged) In class we asserted that if $e_1, \ldots, e_n$ is a basis for V, then the vectors $e_{i_1} \wedge \cdots \wedge e_{i_k}$, where $i_i < \cdots < i_k$ together form a basis for $\bigwedge^k(V)$. Prove this fact.

Problem 3: If V is an n–dimensional vector space and $A : V \to V$ is a linear map, show

(a) there is a well-defined induced map on $\bigwedge^k(V)$ given by

$$v_1 \wedge \cdots \wedge v_k \mapsto A(v_1) \wedge \cdots \wedge A(v_k);$$

(b) the map on $\bigwedge^n(V)$ is simply multiplication by $\det(A)$.

Problem 4: Recall that an element of $\bigwedge^k(V)$ is decomposable if it is of the form $v_1 \wedge \ldots v_k$:

(a) Given four vectors $v, w, x, y \in V$, under what conditions is $v \wedge w + x \wedge y$ a decomposable vector in $\bigwedge^2(V)$?

(b) Show that $\eta \in \bigwedge^2(\mathbb{R}^4)$ is decomposable if and only if $\eta \wedge \eta = 0$.

Problem 5: In the following, M and N are smooth manifolds, $f : M \to N$ is a smooth map, and ω is a differential form on N. Compute $f^*\omega$ in each case.

(a) $M = N = \mathbb{R}^2$;
$f(s,t) = (st, e^t)$;
$\omega = x\,dy - y\,dx$.

(b) $M = \mathbb{R}^2$ and $N = \mathbb{R}^3$;
$f(\theta, \varphi) = ((\cos\varphi + 2)\cos\theta, (\cos\varphi + 2)\sin\theta, \sin\varphi)$;
$\omega = z^2\,dz$.

(c) $M = \{(s,t) \in \mathbb{R}^3 \mid s^2 + t^2 < 1\}$ and $N = \mathbb{R}^3 - \{0\}$;
$f(s,t) = \left(s, t, \sqrt{1 - s^2 - t^2}\right)$;
$\omega = (1 - x^2 - y^2)\,dz$.