For example

1) $(\longleftrightarrow) \looparrowright \mathbb{R}^2$ 

is an immersion, but not an embedding

2) $S^1 \hookrightarrow \mathbb{C}^1$ is an embedding

3) $S^n \hookrightarrow \mathbb{R}^{n+1}$ is an embedding

4) $S^{2n+1} \longrightarrow \mathbb{C}P^n$ is a submersion (also a fiber bundle)

$\mathbb{C}^{n+1} - \{\bar{0}\} \longrightarrow \mathbb{C}P^n$ is too

<u>Theorem</u> (Local Embedding Theorem)
If $f: M \to N$ is smooth, then f is an immersion iff it is a local embedding.

<u>Pf</u> Exercise. ⊠

§ <u>Submanifolds</u>
As we've already seen, its frequently important to underst[and] manifolds which "sit inside" other manifolds.

<u>Defn</u>• An <u>embedded submanifold of M</u> is a subspace S with a smooth structure, with respect to which the inclusion $S \hookrightarrow M$ is a smooth embedding.

• The difference in dimension $\dim(M) - \dim(S)$ is called the <u>codimension</u> of S in M.

Examples

1) Graphs of smooth functions

Let $f : M \to N$ be smooth. The space
$$\Gamma(f) = \{(x,y) \mid x \in M,\ y = f(x)\} \subset M \times N$$
is an embedded submfd.

pf

The map
$$\begin{aligned} \gamma_f : M &\longrightarrow M \times N \\ p &\longmapsto (p, f(p)) \end{aligned}$$
is smooth.

Since
$$\begin{tikzcd} M \arrow[r, "\gamma_f"] \arrow[rr, bend right, "Id"'] & M \times N \arrow[r, "\pi_M"] & M \end{tikzcd}$$
$d\gamma_f$ must be injective at each pt of M.

Thus, γ_f is a smooth immersion, which is a homeomorphism onto its image since π_M is a continuous inverse. $\boxed{}$

2) $S^n \subset \mathbb{R}^{n+1}$ is embedded... graph of
$$(x_1, \ldots, x_n) \longmapsto \left(x_1, \ldots, x_n, \sqrt{1 - |x|^2}\right)$$
for $(x_1, \ldots, x_n) \in \{\text{Ball of radius } 1\}$

3) Appropriate Level Sets of smooth functions $M \xrightarrow{f} N$

Let $f : M \to N$ and $c \in N$. The Level set of f at c is the set $f^{-1}(c)$.

Theorem (Constant rank Level Set theorem)

Let $f: M \to N$ be smooth of constant rank $= r$. Each level set of f is an embedded submfd of codim $= r$.

Pf Let $c \in N$ be given, $S = f^{-1}(c)$, Since f is constant rank, about each point $p \in S$, we can find a charts (u,φ) and (V,ψ) where f looks like

$$f(x^1,..,x^n,x^{n+1},..,x^m) = (x^1,..,x^n,\underbrace{0,..,0}_{n\text{-zeros}}).$$

$$p=(0,..,0) \qquad c=(0,..,0)$$

In this chart, we have

$$S \cap \mathcal{U} = \cancel{\{(0,..,0,x^{n+1},..,x^m)\}}$$
$$= \{(x^1,..,x^n,x^{n+1},..,x^m) \in \mathcal{U} \mid x^1 = .. = x^n = 0\}$$

This means that S satisfies the "local k-slice condition" (see your book), which implies that it is an embedded submfd.

$\blacksquare$

(Idea of local k-slice ... paste things together).

Theorem (Submersion Level Set Theorem)

Let $f: M \to N$ be a smooth submersion. If $p \in N$, the level set $f^{-1}(p)$ is a smooth submfd of M.

$\blacksquare$

We can actually generalize this quite a bit

Def^n Let $f: M \to N$ be smooth. A point $p \in M$ is *regular* if the differential df_p is surjective. It is *critical* otherwise.

Note: 1) If $\dim(M) < \dim(N)$, then $\mathrm{Crit}_f = M$.

Def^n A *regular value* of a smooth map $f: M \to N$ is any point $c \in M$ st $f^{-1}(c)$ consists only of regular points. It is *critical* otherwise.

Note: Any point $p \in N$ missed by f is regular.

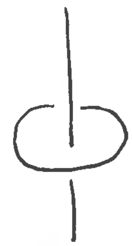
Theorem (Regular Value theorem)
If c is a regular value of $f: M \to N$ (smooth), then $f^{-1}(c)$ is an embedded submfd of M.

Pf
As we showed previously, the set of points $p \in M$ for which $\mathrm{rank}_f(p) = n$ is open in M. Thus it forms a submanifold, Call it $\mathcal{U}$.
Further, by ~~as~~ the def^n of regular value, the subset $f^{-1}(c)$ is contained in this submfd.
If we restrict f to the subset $\mathcal{U}$, then

$$f|_{\mathcal{U}}: \mathcal{U} \to N$$

is a smooth submersion
Thus, by the previous theorem $f^{-1}(c) \subset \mathcal{U} \subset M$ is an embedded submanifold. □

<u>Ex</u>: 1) The map $S^3 \xrightarrow{H} S^2$ is a submersion.
The preimages of points in S^2 look like linked unknots in S^3



2) Let $d(x) = (x^1)^2 + \cdots + (x^n)^2$ be the "distance from the origin squared function" on $\mathbb{R}^n$.

$$d : \mathbb{R}^n \to \mathbb{R}$$

$$d(d)_x = (2x^1, \dots, 2x^n) \Rightarrow \text{All points but } \bar{0} \text{ are regular}$$

$\Rightarrow$ $d^{-1}(r)$ is an embedded submfd for $r \neq 0$.

<u>Remark</u>: Your book contains a discussion of "immersed submanifolds". These are nothing more than 1-1 immersions

<u>Ex</u> Irrationally sloped curve on T^2.

The difference between immersed vs ~~emb~~ embedded submanifolds is whether or not their topology is given by the subspace topology.

<u>Tangent spaces to submanifolds</u>

As we mentioned previously, if $M \subset \mathbb{R}^N$ is an embedded submanifold, then we know how to make sense of

$$T_pM \subset T_p\mathbb{R}^N.$$

its just the set of vectors which are literally tangent to M

More generally, suppose that $S \overset{i}{\hookrightarrow} M$ is an embedded submanifold. Then, at each point $p \in S$, the inclusion gives an injection on the level of tangent spaces

$$di_p : T_pS \longrightarrow T_pM$$

As derivations, we have if $f \in C^\infty(M)$, $v \in T_pS$,

$$i_*(v)(f) = v(f \circ i)$$
$$= v(f|_S)$$

<u>Proposition</u>: Let $S \subset M$ be an embedded submfd, $p \in S$. As a subspace of T_pM, the space T_pS is given by

$$T_pS = \{ v \in T_pM \mid v(f) = 0 \text{ whenever } f \in C^\infty(M) \text{ satisfies } f|_S \equiv 0 \}$$

<u>Proof</u> ($\Rightarrow$) Let $v \in T_pS$, $f \in C^\infty(M)$, $f|_S \equiv 0$.

$$\Rightarrow \quad \text{"}v(f)\text{"} = i_*(v)(f) = v(f \circ i)$$
$$= v(f|_S)$$
$$= v(0)$$
$$= 0_p$$

($\Leftarrow$) Let $v \in T_pM$ and suppose $v(f) = 0$ whenever $f|_S \equiv 0$.

Want a vector $w \in T_pS$ so that
$$v = i_*(w).$$

Let (U, φ) be a coordinate system about p

in which $S \cap \mathcal{U}$ is given by

$$S \cap \mathcal{U} = \{(x^1, \dots, x^n) \in \mathcal{U} \mid x^{k+1} = \dots = x^n = 0\}$$

Then the inclusion is given (in appropriate coords) by

$$(x^1, \dots, x^k) \longmapsto (x^1, \dots, x^k, 0, \dots, 0)$$

In these coords, we have that the subspace

$$T_pS \subset T_pM$$

is spanned by

$$\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^k}.$$

In other words, if $v = v^1 \frac{\partial}{\partial x^1} + \dots + v^n \frac{\partial}{\partial x^n} \in T_pM$, then $v \in T_pS$ iff $v^{k+1} = \dots = v^n = 0$.

Let $f = \varphi(\bar{x}) x^j$, where $\varphi(\bar{x})$ is a bump function with $\operatorname{supp}(\varphi) \subset \mathcal{U}$, extended to be zero on $M - \mathcal{U}$.

Then $f|_S \equiv 0$.

We compute

$$0 = v(f) = v^i \frac{\partial}{\partial x^i}(f) = v^i \underbrace{\left(\frac{\partial}{\partial x^i}\Big|_p \varphi(x) x^j \right)}_{1 \text{ at } 0}$$

$$= v^j$$

$\Rightarrow v \in T_pS.$ ▨

One important example where the above comes up is the boundary of a mfd with boundary.