1

3-manifolds

** Based on a course by John Etnyre at UPenn **

Our goal over the next few months is to discuss some of the basic tools and techniques of 3-manifold topology.

Warning: We will largely omit any discussion of hyperbolic structures on 3-manifolds. This is a very significant issue. Hyperbolic geometry plays an absolutely fundamental role in the study of 3-manifolds.

Examples and Constructions

Basic: $\mathbb{R}^3$, S^3, $B^3 = \{|\vec{x}| \leq 1, x \in \mathbb{R}^3\}$, $T^3 = S^1 \times S^1 \times S^1$,

$\Sigma_g \times I$, $\Sigma_g \times S^1$

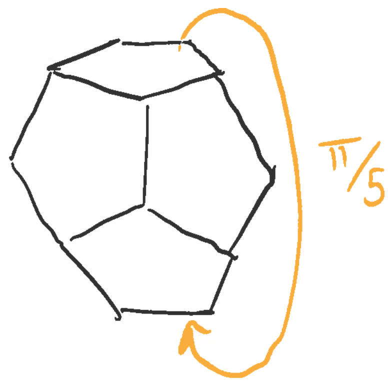
surface of genus $= g$.

Less basic: An identification space obtained from a 3-D polytope.

Eg:

① $T^3 =$ 

② Take a dodecahedron + identify opposite faces via a $\pi/5$ rotation.



This space is called the Poincaré Dodecahedral Space.

__Exercise__ (I expect you all to do this)
Check that this thing is actually a manifold. (Hint: only tricky part is verifying near vertices).

__Fact:__ Any topological space obtained in this way with $\chi(M)=0$ is a 3-manifold.

__Additional constructions__

① Consider

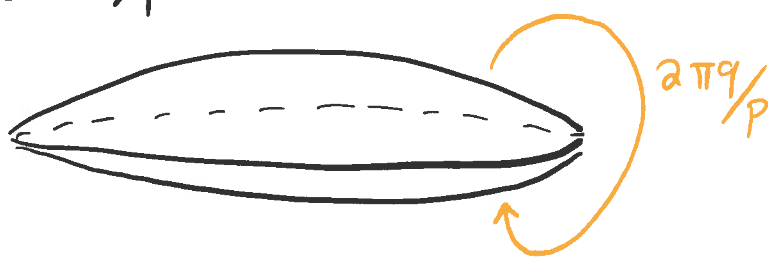
$$S^3 \subset \mathbb{C}^2, \quad S^3 = \{(z,w) \mid |z|^2 + |w|^2 = 1\}$$

Let $S^3 \circlearrowleft \mathbb{Z}/p$ via $(z,w) \longmapsto (e^{2\pi i/p} z, e^{2\pi i q/p} w)$

③ Then we define the lens space

$$L(p,q) := S^3/\mathbb{Z}/p$$

Exercise: We can also check that $L(p,q)$ is obtained from a "lens" via a $2\pi q/p$ rotation

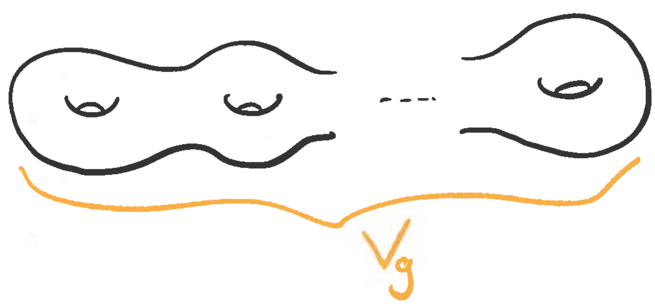


② Heegaard Splittings

Consider a surface Σ_g of genus $= g$.

A handlebody of genus g is a 3-manifold with boundary V_g homeomorphic to Σ_g union its interior in $\mathbb{R}^3$

($\text{ie.}, \partial V_g = \Sigma_g$)



A **Heegaard splitting** of a 3-manifold Y is a decomposition

$$Y = V_1 \cup_{\Sigma} V_1 \, ,$$

where

$$V_1 \cap V_2 = \partial V_1 = \partial V_2 = \Sigma$$

We call Σ the **Heegaard surface** of the splitting and its genus the **Heegaard genus** of the splitting

One can alternatively view Heegaard decomps. as an abstract way of generating 3-mflds.

In other words, let V_1 + V_2 be genus-g handlebodies and

$$h : \partial V_1 \longrightarrow \partial V_2$$

a homeomorphism. Then

$$Y = V_1 \cup_h V_2$$

is a 3-manifold equipped with an explicit Heegaard decomposition.

Note: If h_1 + h_2 are isotopic homeomorphisms, then the 3-manifolds obtained via h_1 and h_2 are homeomorphic.

Theorem: Every closed, oriented 3-manifold has a Heegaard decomposition.

Before we prove this, let's have a brief discussion about exactly what we mean by 3-dimensional topology.

In general, there are 3 categories people work in

<u>Diff</u>: Smooth manifolds up to diffeo

<u>PL</u>: Piecewise linear manifolds up to PL-equivalence

<u>Top</u>: Topological manifolds up to homeomorphism.

In general,

$$\underline{\text{Diff}} \subset \underline{\text{PL}} \subset \underline{\text{Top}}$$

In other words, every smooth manifold has a unique PL structure up to PL-equivalence and determines a unique topological mfd up to homeomorphism.

Milnor won the Fields medal for showing that the 7-sphere supports several smooth structures which are not pairwise diffeomorphic.

⑥

The difference between Diff and Top is most pronounced in dim $= 4$. There, $\underline{\text{Diff}} = \underline{\text{PL}}$, but

Theorem: There are uncountably many pairwise non-diffeomorphic smooth structures on $\mathbb{R}^4$.

However, when $n \leq 3$, we have that

$$\underline{\text{Diff}} = \underline{\text{PL}} = \underline{\text{Top}}$$

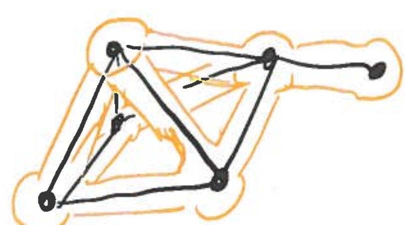
In other words, every topological mfd of dim ≤ 3 has a unique smooth structure

Now, we're ready to prove our theorem.

Proof

Given a 3-manifold Y, let K be a triangulation of Y.

Let V_1 be a tubular neighborhood of the 1-skeleton of Y.



Then V_1 is a handlebody in Y.

⑦

To see that the complement of (the interior of) V_1 is a handlebody, note that it is a neighborhood of the dual 1-skeleton of K.

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