Differential Geometry Homework 6

Problem 1: Let V,W, and Z be vector spaces over $\mathbb{R}$. Show the following:

(a) $V \otimes W$ is naturally isomorphic to $W \otimes V$.
(b) $V \otimes (W \otimes Z)$ is naturally isomorphic to $(V \otimes W) \otimes Z$.
(c) $V^* \otimes W$ is naturally isomorphic to $\mathrm{Hom}(V, W)$.
(d) There are natural isomorphisms $(V \otimes W)^* = V^* \otimes W^* = \mathrm{Hom}(V, W^*) = \mathrm{Hom}(W, V^*)$.
(e) $\mathrm{Hom}(V \otimes W, Y)$ is naturally isomorphic to $\mathrm{Hom}(V, \mathrm{Hom}(W, Z))$.

Problem 2: If V is an n–dimensional vector space and $A : V \to V$ is a linear map, show

(a) there is a well-defined induced map on $\bigwedge^k(V)$ given by

$$v_1 \wedge \cdots \wedge v_k \mapsto A(v_1) \wedge \cdots \wedge A(v_k);$$

(b) the map on $\bigwedge^n(V)$ is simply multiplication by $\det(A)$.

Problem 3: In class we asserted that if $e_1, \ldots, e_n$ is a basis for V, then the vectors $e_{i_1} \wedge \cdots \wedge e_{i_k}$, where $i_i < \cdots < i_k$ together form a basis for $\bigwedge^k(V)$. Prove this fact.

Problem 4: Show that the vectors $v_1, \ldots, v_k \in V$ are linearly independent if and only if $v_1 \wedge \cdots \wedge v_k \neq 0$ in $\bigwedge^k(V)$.

Problem 5: Recall that an element of $\bigwedge^k(V)$ is decomposable if it is of the form $v_1 \wedge \ldots v_k$:

(a) Given four vectors $v, w, x, y \in V$, under what conditions is $v \wedge w + x \wedge y$ a decomposable vector in $\bigwedge^2(V)$?
(b) Show that $\eta \in \bigwedge^2(\mathbb{R}^4)$ is decomposable if and only if $\eta \wedge \eta = 0$.

Problem 6: In the following, M and N are smooth manifolds, $f : M \to N$ is a smooth map, and ω is a differential form on N. Compute $f^*\omega$ in each case.

(a) $M = N = \mathbb{R}^2$;
$f(s, t) = (st, e^t)$;
$\omega = x\,dy - y\,dx$.
(b) $M = \mathbb{R}^2$ and $N = \mathbb{R}^3$;
$f(\theta, \varphi) = ((\cos\varphi + 2)\cos\theta, (\cos\varphi + 2)\sin\theta, \sin\varphi)$;
$\omega = z^2\,dz$.
(c) $M = \{(s, t) \in \mathbb{R}^3 \mid s^2 + t^2 < 1\}$ and $N = \mathbb{R}^3 - \{0\}$;
$f(s, t) = \left(s, t, \sqrt{1 - s^2 - t^2}\right)$;
$\omega = (1 - x^2 - y^2)\,dz$.