# Differential Geometry Homework 5

**Problem 1:** Consider the Heisenberg group

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$$

which is a Lie group diffeomorphic to $\mathbb{R}^3$.

(a) Fine the left-invariant vector fields $X$, $Y$, and $Z$ on $\mathbb{R}^3$ whose values at the identity form the standard basis in $\mathbb{R}^3$.

(b) Compute the Lie brackets $[X, Y]$, $[X, Z]$, and $[Z, Y]$.

**Problem 2:** Show that the Lie groups $S^1 \times \mathrm{SU}(n)$ and $\mathrm{U}(n)$ have the same Lie algebra. Further, show that although $S^1 \times \mathrm{SU}(n)$ and $\mathrm{U}(n)$ are diffeomorphic as smooth manifolds, they are NOT isomorphic as Lie groups.

**Problem 3:** Compute the flow of the following vector fields:

(1) $Z(x, y) = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ on $\mathbb{R}^2$.

(2) $Z(x, y) = (4x - 3y)\frac{\partial}{\partial x} + (6x - 5y)\frac{\partial}{\partial y}$ on $\mathbb{R}^2$.

(3) $Z(x) = (1 + x^2)\frac{\partial}{\partial x}$ on $\mathbb{R}$.

Why are the vector fields in (1) and (2) complete (even before computing the flow)? Draw a picture of each flow.

**Problem 4:** Show that the set of all $m \times n$ matrices of rank $k$ is an embedded submanifold of $\mathbb{R}^{mn}$ of dimension $k(m + n - k)$.

**Problem 5:** Show that $T(x, y, z) = (1 - z^2)y\frac{\partial}{\partial x} - (1 - z^2)x\frac{\partial}{\partial y}$ defines a vector field on $S^2 = \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 = 1\}$ and compute its flow on $S^2$. Can you describe it geometrically?

**Problem 6:** Let $X$ be a vector field on $M$ and $\gamma : I \to M$ an integral curve of $X$. Show that $\gamma$ is the constant map if $\gamma'(t_0) = 0$ for some $t_0 \in I$.