Differential Geometry Homework 4

Problem 1: Let $\mathbb{H}$ denote the space of quaternions. Using the fact that $|pq| = |p||q|$ (please verify this yourself), show that:

(a) S^3 is a Lie group.

(b) The map $q \mapsto [v \mapsto qvq^{-1}]$, where $q \in \mathbb{H}$, $|q| = 1$, and $v \in \text{Im}(\mathbb{H})$ is a Lie group homomorphism from S^3 to $\text{SO}(3)$.

(c) The Lie group $\text{SO}(3)$ is diffeomorphic to $\mathbb{R}P^3$.

Problem 2: Show that the map $f : S^3 \times S^3 \to \text{SO}(4)$ defined by

$$f(q, r) = [v \in \mathbb{H} \mapsto qvr^{-1} \in \mathbb{H}]$$

is a Lie group homomorphism which is onto with kernel $\{\pm(1, 1)\}$. Conclude that $\text{SO}(4)$ is diffeomorphic to $S^3 \times S^3/[(x, y) \sim (-x, -y)]$.

Problem 3: Show that the Stiefel manifold $V(k; n)$ of orthonormal k–frames in $\mathbb{R}^n$, i.e., the set of ordered k–tuples of orthonormal vectors in $\mathbb{R}^n$, is a manifold. What is the dimension of $V(k; n)$?

Problem 4: Let M be a nonempty closed smooth manifold (closed means $\partial M = \emptyset$). Show that there exists no smooth submersion $f : M \to \mathbb{R}^k$ for any $k > 0$.

Problem 5: Suppose $M \subset \mathbb{R}^n$ is an embedded m–dimensional submanifold, and let $UM \subset T\mathbb{R}^n$ be the set of all unit tangent vectors to M:

$$UM = \{(x, v) \in T\mathbb{R}^n \mid x \in M, v \in T_xM, |v| = 1\}.$$

This space is called the *unit tangent bundle of M*. Prove that UM is an embedded $(2m - 1)$–dimensional submanifold of $T\mathbb{R}^n$.

(As you probably suspect, UM is more naturally an embedded submanifold of TM. The reason for using $T\mathbb{R}^n$ as opposed to TM is that it allows us to talk about the "length" of a tangent vector without first defining Riemannian metrics or using other round-about methods.)