Differential Geometry Homework 2

Problem 1: Show that $O(n)$ and $U(n)$ are manifolds. Show that $U(n)$ is connected and that $O(n)$ has two components. (Recall that $A \in O(n)$ if $AA^T = I$ and $A \in U(n)$ if $A\overline{A}^T = I$.)

Problem 2: Consider the map $f : \mathbb{R}P^2 \to \mathbb{R}^3$ given by

$$[x : y : z] \mapsto (yz, xz, xy).$$

Show that f is an immersion except at six points. (In this problem, we're viewing $[x : y : z]$ as the equivalence class of points on S^2 related by the antipotal map.)

Problem 3: Let $f : \mathbb{R}P^2 \to \mathbb{R}^6$ be given by

$$[x : y : z] \mapsto (x^2, y^2, z^2, \sqrt{2}xy, \sqrt{2}xz, \sqrt{2}yz).$$

(a) Show that f is an embedding of $\mathbb{R}P^2$ into $\mathbb{R}^6$

(b) Let

$$V = \{(x_1, \ldots, x_6) \in \mathbb{R}^6 \,|\, x_1 + x_2 + x_3 = 1\}.$$

Show that $f(\mathbb{R}P^2) \subset V \cap S^5(1) = S^4(\sqrt{2/3})$.

Problem 4: Let $A : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Show that the map $A_* : T_x\mathbb{R}^n \to T_{Ax}\mathbb{R}^m$ is equal to A for each point $x \in \mathbb{R}^n$. Also, think about how this problem relates to the change of variables formula for multidimensional integrals (no need to write anything up about this.).

Problem 5: Show that if M and N are smooth, then $T(M \times N)$ is diffeomorphic to the product $TM \times TN$.