Differential Geometry Homework 1

Problem 1: Let X be the set of all points $(x, y) \in \mathbb{R}^2$ such that $y = \pm 1$, and let M be the quotient of X by the equivalence relation generated by $(x, -1) \sim (x, 1)$ for all $x \neq 0$. Show that M is locally Euclidean and second countable, but not Hausdorff. (This space is called the *line with two origins*.)

Problem 2: Show that the disjoint union of uncountably many copies of $\mathbb{R}$ is locally Euclidean and Hausdorff, but not second countable.

Problem 3: *Complex projective n–space*, denoted $\mathbb{C}P^n$, is the set of 1–dimensional complex-linear subspaces of $\mathbb{C}^{n+1}$, with the quotient topology inherited form the natural projection $\pi : \mathbb{C}^{n+1} - \{0\} \to \mathbb{C}P^n$. Show that $\mathbb{C}P^n$ is a compact $2n$–dimensional topological manifold, and show how to give it a smooth structure analogous to the one discussed in Lee for $\mathbb{R}P^n$. (Note: We identify $\mathbb{C}^{n+1}$ with $\mathbb{R}^{2n+2}$ via $(x^1 + iy^1, \ldots, x^{n+1} + iy^{n+1}) \leftrightarrow (x^1, y^1, \ldots, x^{n+1}, y^{n+1})$.)

Problem 4: Prove that if $F : M \to N$ is a smooth map between smooth manifolds, then the coordinate representation of F is smooth with respect to *any* pair of charts on M and N.

Problem 5: Compute the coordinate representations for each of the following maps in stereographic coordinates, and use this to prove that each map is smooth.

(1) For each $n \in \mathbb{Z}$, the n^{th}*-power map* $p_n : S^1 \to S^1$, given in complex notation by $p_n(z) = z^n$.
(2) The antipotal map, $\alpha : S^n \to S^n$, given by $\alpha(x) = -x$.
(3) The Hopf fibration $H : S^3 \to S^2$, given by $H(z, w) = (z\overline{w} + w\overline{z}, iw\overline{z} - iz\overline{w}, z\overline{z} - w\overline{w})$, where we think of $S^3 \subset \mathbb{C}^2$ as the subset $\{(w, z) | \, |w|^2 + |z|^2 = 1\}$.

Problem 6: Let M be a connected smooth manifold, and let $\pi : \widetilde{M} \to M$ be a topological covering map. Show that there is only one smooth structure on $\widetilde{M}$ such that π is a smooth covering map.

Problem 7: Let $\mathbb{C}P^n$ be complex projective n–space as defined above.

(1) Show that the quotient map $\pi : \mathbb{C}^{n+1} - \{0\} \to \mathbb{C}P^n$ is smooth.
(2) Show that $\mathbb{C}P^1$ is diffeomorphic to S^2.