Goal: To get a firm grip on the basics of smooth manifold theory.

Main Topics: Transversality, Lie groups, de Rham cohomology / Hodge theorem

We will focus on Lie groups + de Rham cohomology.

Warning: This course will be fast paced and you are expected to keep up. Despite the course number, this is certainly "first-year" stuff.

Book(s):
1) Intro to Smooth Manifolds – Lee
2) Differential Topology – Guillemin + Pollack
3) Differential Geometry – Spivak
4) Foundations of Differentiable Manifolds + Lie Groups – Warner.

Topological Preliminaries

Def^n A topological space X is called paracompact If every open cover of X has a locally finite refinement.

Def^n X is
1) 2nd countable $\Rightarrow$ countable basis for the topology of X
2) Hausdorff $\Rightarrow$ points can be separated by open sets
3) Regular $\Rightarrow$ pts can be separated from closed sets by open sets
4) Normal $\Rightarrow$ Closed sets can be separated by open sets.

Although we don't need all of these, the following implications are true.

Theorem:
1) Regular + 2nd countable $\Rightarrow$ metrizable
2) Metric Space $\Rightarrow$ Paracompact (Stone's Theorem)
3) Regular + 2nd countable $\Rightarrow$ Normal

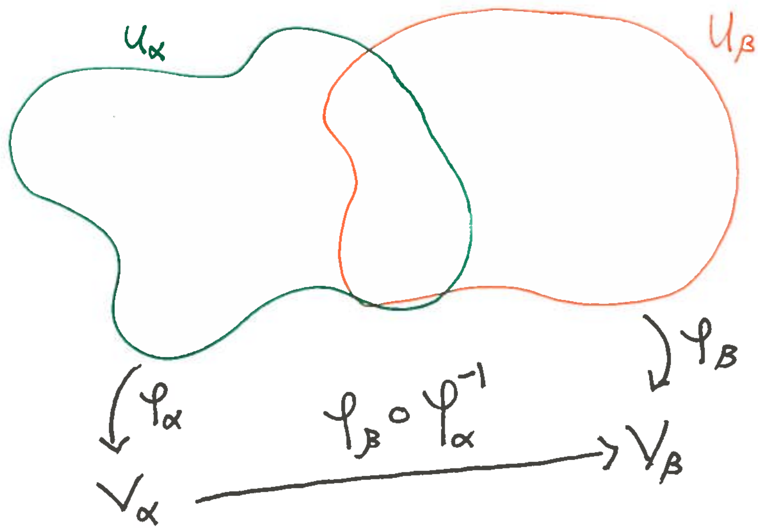
4) Paracompact + Locally Compact + Connected $\Rightarrow$ 2nd countable
5) Paracompact $\Rightarrow$ Normal
6) If X is Hausdorff, $\{\mathcal{U}_n\}_{n\geq 1}^{\infty}$ an open cover of X st
   1) $\overline{\mathcal{U}_n} \subset \mathcal{U}_{n+1}$
   2) $\overline{\mathcal{U}_n}$ compact

   Then X is Paracompact.

<u>Def^n</u> A <u>manifold</u> is a paracompact topological space M such that each point in M has an neighborhood homeomorphic to an open subset $\mathcal{U} \subset \mathbb{R}^n$

- A <u>C^k-Atlas</u> for a manifold M is a collection $\{(\mathcal{U}_\alpha, \varphi_\alpha)\}$ of pairs st
  - $\{\mathcal{U}_\alpha\}$ is an open cover of M, $\bigcup \mathcal{U}_\alpha = M$
  - $\{\varphi_\alpha\}$ are charts, $\varphi_\alpha : \mathcal{U}_\alpha \longrightarrow V_\alpha \subset \mathbb{R}^n$

  Further, if $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \neq \emptyset$, then



$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(\mathcal{U}_\alpha \cup \mathcal{U}_\beta) \longrightarrow \varphi_\beta(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$$

is a C^k-diffeomorphism.

· We call the compositions "$\varphi_\alpha \circ \varphi_\beta^{-1}$" <u>transition</u> functions.

If the transition functions are of class C^∞, we simply call M a <u>smooth manifold</u>.

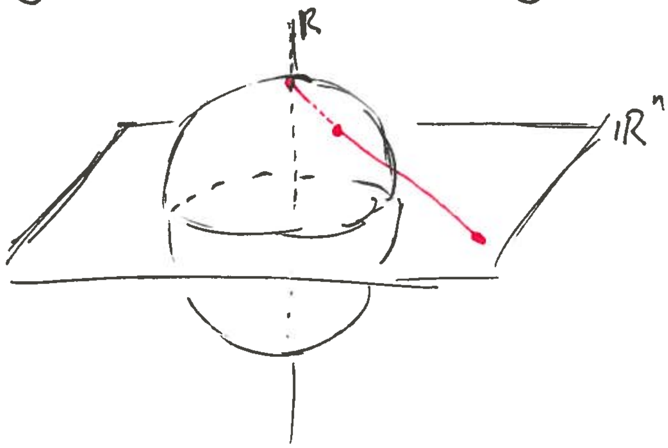
Def^n A C^k-atlas is called <u>maximal</u> if it is not contained in any larger C^k-atlas.

<u>heorem</u>: Let M be a manifold with a maximal C^k-atlas $\{(U_\alpha, \varphi_\alpha)\}$. Then $\exists$ a sub-atlas of class C^∞ or C^ω.

<u>xamples</u>:

1) $\mathbb{R}^n$ with atlas $\{(\mathbb{R}^n, Id)\}$

2) S^n with charts given by Stereographic projection.



$$\varphi_N : S^n - \{N\} \longrightarrow \mathbb{R}^n$$
$$(x_1, \dots, x_{n+1}) \longmapsto \left(\frac{x_1}{1-x_{n+1}}, \dots, \frac{x_n}{1-x_{n+1}}\right)$$

$$\varphi_N^{-1} : \mathbb{R}^n \longrightarrow S^n - \{N\}$$
$$(x_1, \dots, x_n) \longmapsto \frac{1}{1+\|x\|^2}(2x_1, \dots, 2x_n, \|x\|^2 - 1)$$

$$\varphi_S : S^n - \{S\} \longrightarrow \mathbb{R}^n$$
$$(x_1, \dots, x_{n+1}) \longmapsto \left(\frac{x_1}{1+x_{n+1}}, \dots, \frac{x_n}{1+x_{n+1}}\right)$$

$$\varphi_S^{-1} : \mathbb{R}^n \longrightarrow S^n - \{S\}$$
$$(x_1, \dots, x_n) \longmapsto \frac{1}{1+\|x\|^2}(2x_1, \dots, 2x_n, 1-\|x\|^2$$

Thus,

$$\varphi_N \circ \varphi_S^{-1} : \mathbb{R}^n - \{\bar{0}\} \to \mathbb{R}^n - \{\bar{0}\}$$
$$(x_1, \dots, x_n) \longmapsto \frac{1}{\|x\|}(x_1, \dots, x_n)$$

imilarly

$$\varphi_S \circ \varphi_N^{-1} : \mathbb{R}^n - \{\bar{0}\} \to \mathbb{R}^n - \{\bar{0}\}$$
$$(x_1, \dots, x_n) \longmapsto \frac{1}{\|x\|}(x_1, \dots, x_n)$$

3) Any open subset of a smooth manifold is itself a smooth manifold

4) T^2, the 2-torus $\mathbb{R}^2/\mathbb{Z}^2$ (<u>What do the transition maps look like?</u>

5) A genus g surface $\Sigma_g = \mathbb{H}^2/\Gamma$. Not the same $\mathbb{H}$

6) Projective spaces: $\mathbb{R}P^n$, $\mathbb{C}P^n$, $\mathbb{H}P^n$

Equivalence classes of vectors in $\mathbb{R}^{n+1}-\{0\}$, $\mathbb{C}^{n+1}-\{\bar{0}\}$, $\mathbb{H}^{n+1}-\{\bar{0}\}$ up to scalar mult.

Our next example is very important in geometry and topology.

7) $G_k(V)$, the Grassman manifold of k-planes in V

<u>Note:</u> In the literature, people frequently do not specify whether or not the k-planes are <u>oriented</u> or not. It matters. Since we do not yet understand orientations, we delay the discussion. and assume our planes are <u>unoriented</u>.

<u>Proof:</u> Let P, Q be subspaces of V of complementary dimension k, $(n-k)$ st $V = P \oplus Q$.

- If $A: P \to Q$ is linear, then the graph
$$\Gamma(A) = \{x + Ax \mid x \in P\}$$
is a subspace of V with $\Gamma(A) \cap Q = \{\bar{0}\}$.
- Similarly, any subspace Γ of $\dim = k$, with $\Gamma \cap Q = \{\bar{0}\}$ is $\Gamma(A)$ for some $A: P \to Q$.

Let
- $L(P,Q)$ denote the space of linear maps $P \to Q$.
- $\mathcal{U}_Q$ = subset of $G_k(V)$ of subspaces having trivial intersection with Q.

(5)

- From the discussion, we have that

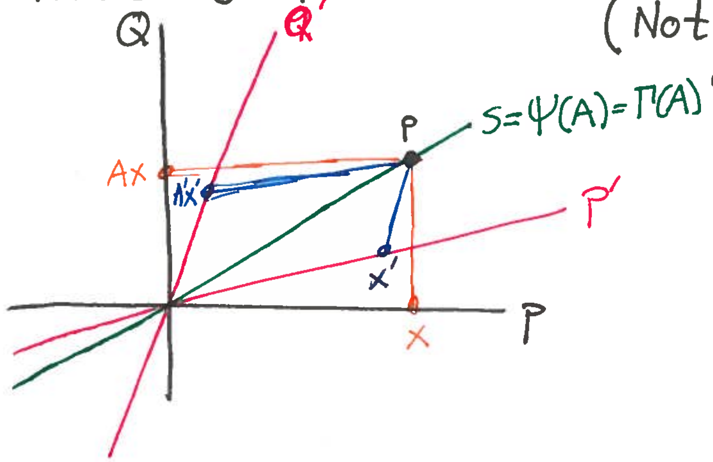
$$\begin{aligned}\psi : L(P,Q) &\longrightarrow \mathcal{U}_Q \\ A &\longmapsto \Gamma(A)\end{aligned}$$

is a bijection.

- Denote its inverse by $\varphi : \mathcal{U}_Q \to L(P,Q) = \mathcal{M}((n-k)\times k, \mathbb{R})$ $\mathbb{R}^{(n-k)\times k}$ // and declare this to be a coord chart. choose a basis for $P+Q$

- Let $\mathcal{U}_{Q'}$ denote an alternate such chart corresponding to the pair (P',Q').

Then $\varphi(\mathcal{U}_Q \cap \mathcal{U}_{Q'}) \subset L(P,Q)$ is the set of all $A \in L(P,Q)$ whose graphs intersect Q' trivially. (Note, this is an open condition).



Now, we simply need to show that the transition maps are smooth.

Let $A \in \varphi(\mathcal{U}_Q \cap \mathcal{U}_{Q'}) \subset L(P,Q)$ be given, and let S denote $\psi(A) = \Gamma(A) \subset V$.

Let $A' = \varphi' \circ \psi(A)$. By def^n, A' is the ! map $A' : P' \to Q'$ whose graph is S.

If $p \in S$, then $\begin{aligned} p &= x + Ax \text{ for a ! } x \in P, \\ p &= x' + A'x' \text{ for a ! } x' \in P'. \end{aligned}$

Let $I_A : P \to V$ be the map $x \mapsto x + Ax$,
$\pi_{P'} : V \to P'$ the quotient w/ kernel Q'.

The composition is an isomorphism (since $S \cap Q = S \cap Q' = \{$

Observe:

$$x + Ax - x' = A'x' \in Q'.$$

Thus,

$$0 = \pi_{P'}(x + Ax - x')$$
$$= (\pi_{P'} \circ I_A)(x) - x'$$

In other words,

$$x' = (\pi_{P'} \circ I_A)(x)$$

and

$$A'x' = I_A x - x' = I_A \circ (\pi_{P'} \circ I_A)^{-1}(x') - x'$$

Since $I_A, (\pi_{P'} \circ I_A)^{-1}$ depend smoothly on the entries of A, so too must the entries of A'.

<u>Exercise</u>: Check the 2^{nd} countability + Hausdorff conditions. ☒

To obtain a wealth of examples using tools we don't quite yet understand, I simply state

<u>Theorem</u>: Let M be a manifold and G a (Lie) group which acts smoothly, freely and properly on M, then the quotient space M/G is a manifold admitting a canonical smooth structure.

<u>Note</u>: This is one way to see that the projective spaces $\mathbb{R}P^n, \mathbb{C}P^N, \mathbb{H}P^N$ are manifolds.

$$\mathbb{R}P^n = S^n/\mathbb{Z}_2 = \mathbb{R}^{n+1} - \{\vec{0}\} / \mathbb{R}^*$$
$$\mathbb{C}P^n = S^{2n+1}/S^1 = \mathbb{C}^{n+1} - \{\vec{0}\} / \mathbb{C}^*$$
$$\mathbb{H}P^n = \mathbb{H}^{n+1} - \{\vec{0}\} / \mathbb{H}^*$$