§3 Submersions, Immersions and Embeddings

OK, now back to smooth maps between manifolds

Def^{n} Let $f: M \to N$ be smooth, $p \in M$. The rank of f at p is the rank of the linear map $df_p: T_pM \to T_{f(p)}N$

We call a map $f: M \to N$ a

- Submersion if $\mathrm{rank}(f_p) = \dim(N)$ $\forall\, p \in M$
- Immersion if $\mathrm{rank}(f_p) = \dim(M)$ $\forall\, p \in M$

Proposition: Let $f: M \to N$ be smooth and $p \in M$. If df_p is surjective, then $\exists$ a nbhd U of p st f is a submersion on U. Similarly, if df_p is injective, $\exists$ a nbhd U of p st f is an immersion on U.

Pf The hypothesis implies that in small coordinate nbhds of p and $f(p)$, the diff'l

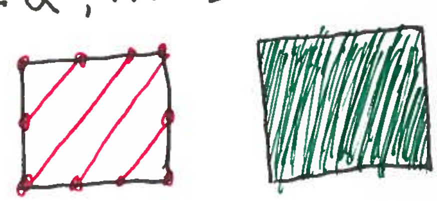
$$df_p \in M(n \times m)$$

has full rank.

This is an open condition, so df_q has full rank for $q \in \tilde{U}_p \subseteq U$ a (potentially) smaller nbhd of p. ∎

Examples:

1) $S^1 \hookrightarrow \mathbb{C}^1 \cong \mathbb{R}^2$
2) $M_i \hookrightarrow M_1 \times M_2 \times \cdots \times M_n$
3) $\pi_i: M_1 \times \cdots \times M_n \longrightarrow M_i$
4) $\pi: TM \longrightarrow M$
5) $\mathbb{R} \longrightarrow T^2$ as a curve of slope
   - p/q
   - α, irrational



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To continue on, we need a really important theorem

Theorem: (Inverse FT) Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be smooth. If $p \in \mathbb{R}^n$ has df_p invertible, then $\exists$ a nbhd $\mathcal{U}$ of p on which f is a local diffeo.

This theorem is not too difficult to prove, and you can look in the appendix for details.

From the Inverse FT, we obtain

Theorem (Inverse FT for manifolds)
Let $f : M \to N$ be smooth. If $p \in M$ is a point where df_p is invertible, then $\exists$ connected nbhds $\mathcal{U}_0$ of p + V_0 of $f(p)$ st $f : \mathcal{U}_0 \to V_0$ is a local diffeo.

Pf
- df_p an isom $\Rightarrow \dim(M) = \dim(N)$
- Let (U, φ) be a chart about p and (V, ψ) a chart about $f(p)$ st $f(U) \subset V$.
- φ and ψ diffeos $\Rightarrow$

$$d\tilde{f}_0 = d\psi_{F(p)} \circ dF_p \circ d\varphi_0^{-1}, \quad \text{nonsingular}$$

where

$$\tilde{f} = \psi \circ f \circ \varphi^{-1} : \varphi(U) \to \psi(V)$$

- By the Inverse FT, $\exists$ opens $\tilde{\mathcal{U}}_0 \subset \varphi(U)$ and $\tilde{V}_0 \subset \psi(V)$ st $\tilde{f} : \tilde{\mathcal{U}}_0 \to \tilde{V}_0$ is a local diffeo.

Thus,

$$\tilde{f} : \mathcal{U}_0 \to V_0$$

is a local diffeo, where $\mathcal{U}_0 = \varphi^{-1}(\tilde{\mathcal{U}}_0)$, $V_0 = \psi^{-1}(\tilde{V}_0)$

$\square$

<u>Theorem</u>: Let $f: M \to N$ be smooth of constant rank $= r$. For each $p \in M$, $\exists$ charts (U, φ) of p and (V, ψ) of $f(p)$ st

1) $f(U) \subset V$
2) In coordinates,

$$F(x^1, \ldots, x^r, x^{r+1}, \ldots, x^m) = (x^1, \ldots, x^r, 0, \ldots, 0)$$

<u>Pf</u>: As before, we may assume we're working in $\mathbb{R}^m$ and $\mathbb{R}^n$ since the thm is local

- Translate so $p = \bar{0}$, $f(p) = \bar{0}$.
- $\mathrm{rank}(f) = r \Rightarrow \exists$ an $(r \times r)$-submatrix of df_p ~~so~~ which is non-singular.
- Reorder coordinates so this submatrix is in the upper left.
- Relabel coordinates as

$$(\bar{x}, \bar{y}) = (x^1, \ldots, x^r, y^1, \ldots, y^{m-r}) \text{ in } \mathbb{R}^m$$
$$(\bar{v}, \bar{w}) = (v^1, \ldots, v^r, w^1, \ldots, w^{n-r}) \text{ in } \mathbb{R}^n$$

In these coordinates

$$f(\bar{x}, \bar{y}) = (Q(\bar{x}, \bar{y}), R(\bar{x}, \bar{y}))$$

for smooth maps

$$Q : U \to \mathbb{R}^r$$
$$R : U \to \mathbb{R}^{n-r}$$

By assumption

$$\det\left(\frac{\partial Q^i}{\partial x^j}\right) \neq 0 \quad \text{at } (\bar{x}, \bar{y}) = \bar{0}.$$

- Define $\varphi : U \to \mathbb{R}^m$ by

$$\varphi(\bar{x}, \bar{y}) = (Q(\bar{x}, \bar{y}), \bar{y})$$

Then

$$d\varphi|_{(\bar{0},\bar{0})} = \begin{pmatrix} \frac{\partial Q^i}{\partial x^j}(0,0) & \frac{\partial Q^i}{\partial y^j}(0,0) \\ 0 & I \end{pmatrix}$$

is non-singular.

By the IFT, $\exists$ connected nbhds $\mathcal{U}_0$ of $(\bar{0},\bar{0})$ and $\tilde{\mathcal{U}}_0$ of $\varphi(\bar{0},\bar{0}) = (\bar{0},\bar{0})$ st

$$\varphi : \mathcal{U}_0 \longrightarrow \tilde{\mathcal{U}}_0$$

is a diffeo.

- By shrinking, we may assume $\tilde{\mathcal{U}}_0$ is a cube (open).
- Let $\varphi^{-1}(\bar{x},\bar{y}) = (A(\bar{x},\bar{y}), B(\bar{x},\bar{y}))$

It follows that

$$(\bar{x},\bar{y}) = \varphi \circ \varphi^{-1}(\bar{x},\bar{y}) = \big(Q(A(\bar{x},\bar{y}), B(\bar{x},\bar{y})),\ B(\bar{x},\bar{y})\big)$$

$$\Rightarrow B(\bar{x},\bar{y}) = \bar{y},$$
$$\Rightarrow Q(A(\bar{x},\bar{y}),\bar{y}) = \bar{x}$$

Also,

$$F \circ \varphi^{-1}(\bar{x},\bar{y}) = F(A(\bar{x},\bar{y}),\bar{y}) = \big(Q(A(\bar{x},\bar{y}),\bar{y}),\ R(A(\bar{x},\bar{y}),\bar{y})\big)$$
$$= (\bar{x}, \tilde{R}(\bar{x},\bar{y})).$$

Observe that since φ^{-1} is a diffeo, this map must have rank = rank$(f) = r$.

- Computing the derivative

$$df|_{(0,0)} = \begin{pmatrix} \underbrace{I}_{(r\times r)} & 0 \\ \frac{\partial \tilde{R}^i}{\partial x^j}\Big|_{(0,0)} & \frac{\partial \tilde{R}^i}{\partial y^j}\Big|_{(0,0)} \end{pmatrix}$$

For this matrix to have rank $= r$,

$$\left(\frac{\partial \tilde{R}^i}{\partial y^j}(0,0)\right) \equiv \bar{0}$$

Thus, $\tilde{R}$ is independent of $\bar{y}$.

- Let

$$S(x) = \tilde{R}(x,0)$$

$$F \circ \varphi^{-1}(\bar{x},\bar{y}) = (x, S(x))$$

To finish things off, we need a new smooth chart for $(\bar{0},\bar{0}) \in V$.

- Let $V_0 = \{(v,w) \in V \mid (v,0) \in \tilde{U}_0\}$

Since $\tilde{U}_0$ is a cube, and since

$$F \circ \varphi^{-1}(\bar{x},\bar{y}) = (x, S(x)),$$

it follows that

$$F \circ \varphi^{-1}(\tilde{U}_0) \subset V_0$$

and thus that

$$F(U_0) \subset V_0$$

- Define $\psi : V_0 \to \mathbb{R}^n$ by

$$\psi(\bar{v},\bar{w}) = (v, w - S(v))$$

Then (V_0, ψ) is a smooth chart.

It follows that

$$\psi \circ F \circ \varphi^{-1}(\bar{x},\bar{y}) = \psi(\bar{x}, S(\bar{x})) = (\bar{x}, S(\bar{x}) - S(\bar{x})) = (\bar{x}, \bar{0})$$

▨

Corollary: • If f is a submersion, then $\exists$ coordinates in which
$$\tilde{f}(x^1,\dots,x^n,x^{n+1},\dots,x^m) = (x^1,\dots,x^n)$$
• If f is an immersion, then $\exists$ coordinates in which
$$\tilde{f}(x^1,\dots,x^m) = (x^1,\dots,x^m,0,\dots,0)$$

Corollary: Let $f: M \to N$ be smooth and M connected. TFAE
1) $\forall\, p \in M$ $\exists$ smooth charts containing p & $f(p)$ in which the coordinate reps. of f is linear.
2) f has constant rank.

pf: (1) $\Rightarrow$ (2)
Use the coord reps to compute the rank. It can't change on a connected comp (like analytic continuation).
use path-connectedness

(2) $\Rightarrow$ (1)
This is the rank theorem. $\boxed{\equiv}$

Theorem (Global rank theorem)
Let $f: M \to N$ be a smooth map of constant rank.
1) If f is surjective, then f is a submersion.
2) If f is injective, then f is an immersion
3) If f is bijective, then f is a diffeo.

pf
(1) Suppose $\underset{r}{\text{rank}(f)} < \underset{n}{\dim(N)}$
Then, by the rank thm, about every pt $p \in M$, $\exists$ coords (U,φ) and (V,ψ) of $f(p)$ st f looks like
$$\tilde{f}(x^1,\dots,x^r,x^{r+1},\dots,x^m) = (x^1,\dots,x^r,\underbrace{0,\dots,0}_{n-r \text{ zeros}})$$

In particular
$\tilde{f}(\varphi(U))$,
and $\therefore f(U)$ are nowhere dense subsets of $\mathbb{R}^n$ and N respectively.

Now, cover M by countably many such coordinate charts.

By the assumption, we have

$$N = f(M) = \bigcup_{i=1}^{\infty} f(U_i)$$

- However, this would mean that N is expressible as the countable union of nowhere dense subsets.
- By the Baire category theorem, this is impossible ✱

(2) If f is not a smooth immersion, $r = \operatorname{rank}(f) < \dim(M) = n$

So, $\exists$ charts in which

$$\hat{f}(x^1, \dots, x^r, x^{r+1}, \dots, x^n) = (x^1, \dots, x^r, 0, \dots, 0)$$

The pts $(\bar{0}, 0)$ and, $(\bar{0}, \varepsilon, \bar{0})$ are both sent to $\bar{0}$ ✱

(3) By (1), (2), f is an immersion and submersion. Thus, it is a local diffeo, and, in turn, a diffeo by bijectivity ▨

I want to draw your attention to the distinction between immersions and embeddings.

<u>Def^n</u> A map $f: M \to N$ is a <u>smooth embedding</u> if it is an immersion and f is a homeomorphism of f onto its image $f(M) \subset N$.