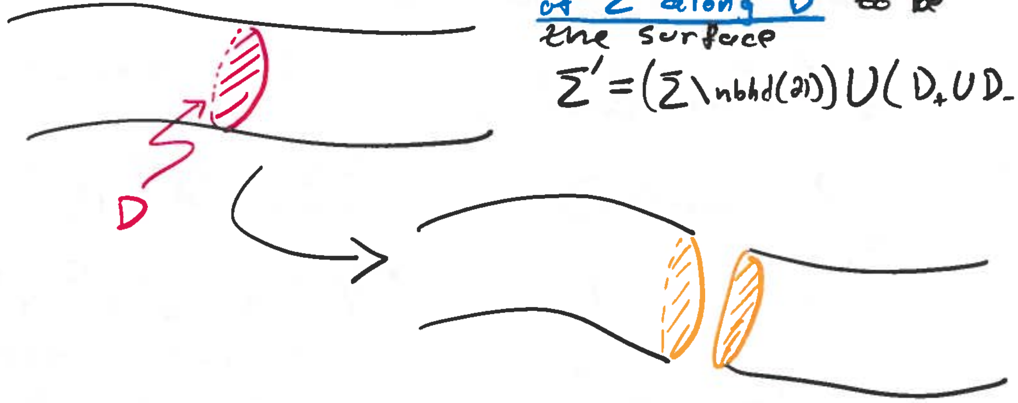
Our goal now is to prove Lemma (E) from the previous lecture.

To do this, we need to understand a few facts about spheres in 3-manifolds.

Suppose that $\Sigma \subset Y$ is a surface and $D \subset Y$ is an embedded disk for which $D \cap \Sigma = \partial D \cap \Sigma \cong S^1$. We define **surgery of Σ along D** to be the surface

$$\Sigma' = (\Sigma \setminus \mathrm{nbhd}(\partial D)) \cup (D_+ \cup D_-)$$

pic



Lemma (F) Suppose $S \subset Y$ is a 2-sphere and that $\tilde{S} = S' \amalg S''$ is the result of surgery of S along a disk D. Then if S' and S'' both bound balls in Y, so does S.

Proof: There are two cases

(1)

$\Rightarrow S$ bounds $B' \cup_D B''$

(2)

In this case, $B' \subset B''$ and S bounds $\overline{B'' \setminus B'}$

$\boxtimes$

②

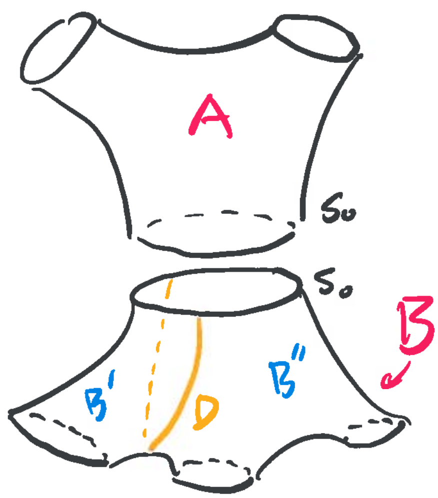
Lemma (G): Let S be an independent system of 2-spheres in Y and $S_0 \in S$ one of these spheres.
Suppose that $S_0' \amalg S_0''$ is obtained from S_0 by surgery along a disk D. Then one or both of

$$(S \setminus S_0) \amalg S_0' \quad \text{or} \quad (S \setminus S_0) \amalg S_0''$$

is an independent system.

Proof: Let A and B denote the components of $Y \setminus\setminus S$ that meet at S_0.

Pic



Note that we could have $A = B$.

Assume W/LOG that $D \subset B$, and that B' and B'' are the components of $B \setminus\setminus D$ (Note that $B' = B''$, $B' \cong Y$, etc. are possible)

Now, since S is indep, if $Y \setminus\setminus S'$ contains a punctured S^3 component, then that component must contain S_0'.

Let $N = (A \cup B) \cup_\partial (3\text{-balls})$. This is a compact manifold.

If S' is not independent, then S_0' bounds a ball in N.
Similarly, if S'' is not independent, then S_0'' bounds a ball in N.

Thus, if both S' and S'' are not independent, S_0 bounds a ball in N (by lemma (G)).

∎

③

We're now ready to prove Lemma Ⓔ

<u>proof</u> (Lemma Ⓔ)

Fix a triangulation T of Y and let S be an indep. system of k 2-spheres in Y.

Isotope S so as to minimize its weight with respect to T.

$$w(S) = \#(S \cap T^{(0)})$$

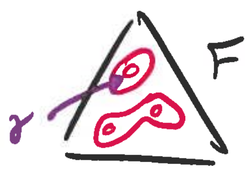
Recall that we previously proved that if weight is minimized, then the intersections of S with $\partial\tau$ (τ a 3-simplex) look like

<u>Claim</u>① We can assume no 0-gons.

Consider a face F of a 3-simplex that contains a 0-gon

Let γ be an innermost 0-gon on F.

<u>pic</u>



Since γ is innermost, it bounds a disk in F which is disjoint from S (except at γ).

Surger S along this disk.

We obtain two new systems of k 2-spheres, S' and S'', at least one of which is indep. by Lemma Ⓖ.

Now, rince + repeat to remove the remaining 0-gons.

<u>Claim</u>② For each 3-simplex $\tau \in T$, we can assume each component of $S \cap \tau$ is a disk.

④

Suppose that τ is a 3-simplex which intersects S in something other than a collection of disks.

pic

First, observe that no component of S can be entirely contained in τ since τ is irreducible.

Thus any of these non-disk components are homeo to punctured 2-spheres.

Let P be an innermost non-disk component

In this case, $\exists$ a disk $D \subset \tau$ st

$$D \cap S = D \cap P = \partial D \cap P$$

and ∂D is isotopic in P to a comp. of ∂P.

Surger S along D to get $S' + S''$. One of these will be independent + both satisfy

$$w(S'), w(S'') < w(S)$$

$\boxtimes$