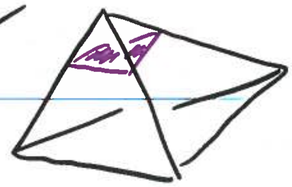
1

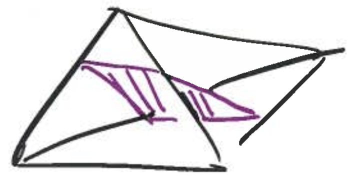
Recall that in the previous lecture we finished with...

Def^{n} A surface Σ in a triangulated 3-manifold Y is called **normal** if for each 3-simplex $\tau \in T$, the components of $\Sigma \cap \tau$ have the following forms

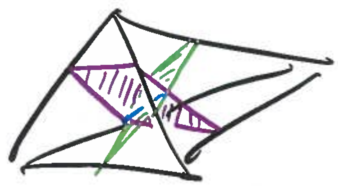
Triangles



Squares



Observe that for each tetrahedron, we can have at most one type of square intersection



Theorem: Let T be a triangulation of a 3-manifold Y with $t = \#(\text{3-simplices of } T)$. If $\Sigma \subset Y$ is an embedded normal surface with at least

$$6t + \dim H_1(Y; \mathbb{Z}/2)$$

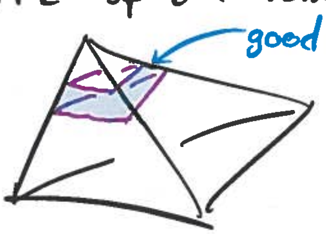
components. Then Σ has a pair of parallel components.

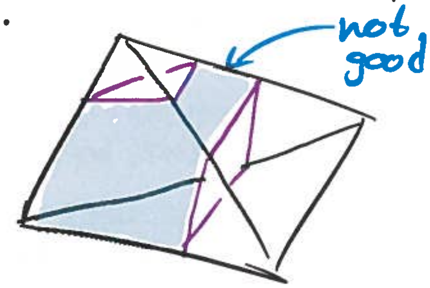
To prove this theorem, we need a definition and a few lemmas.

Def^{n} Let $\tau \in T$ be a 3-simplex. A component of

$$\tau \backslash\backslash \Sigma := \tau \setminus \mathrm{nbhd}(\Sigma)$$

is **good** if the region lies between two components of $\Sigma \cap \tau$ of the same type.



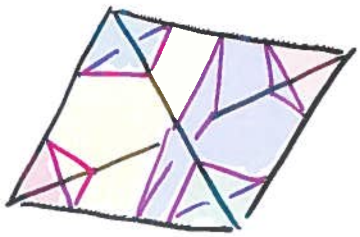


②

Lemma (A): At most six components of $\tau \setminus\setminus \Sigma$ are bad

Proof:

PIC



All current regions are bad.

Adding any additional components to $\Sigma \cap \tau$ just creates product regions.

$\boxtimes$

Def: A component X of $Y \setminus\setminus \Sigma$ is **good** if $X \cap \tau$ is good for all $\tau \in T$.

Def: An **I-bundle** over Σ is a space E and a map $p: E \to \Sigma$ st for each $x \in \Sigma$, $\exists$ a nbhd $\mathcal{U}$ of x with

$$\begin{tikzcd} p^{-1}(\mathcal{U}) \arrow[rr, "\cong"] \arrow[dr, "p"'] & \arrow[d, phantom, "\circ"] & \mathcal{U} \times I \arrow[dl, "\pi"] \\ & \mathcal{U} & \end{tikzcd}$$

Observe that $p: \partial E \to \Sigma$ is a 2-fold covering. There are two possibilities for this cover

① It's trivial and $\partial E \cong \Sigma \amalg \Sigma$

$\implies E \cong \Sigma \times I$

② ∂E is connected and E is a **twisted** I-bundle

Think Möbius band

Lemma (B): Suppose $\Sigma \subset Y$ is normal. Then every good component of $Y \setminus\setminus \Sigma$ is an I-bundle.

Proof: Locally you have a product structure... glue them together

$\boxtimes$

③

Lemma ⓒ: Let E be an I-bundle over Σ. Then
$$H_1(E,\partial E;\mathbb{Z}/2)\cong \mathbb{Z}/2$$

Proof

$$H_1(E,\partial E;\mathbb{Z}/2)\underset{PD}{\cong} H^2(E;\mathbb{Z}/2)\overset{UCT}{\cong} H_2(E;\mathbb{Z}/2)$$
$$\cong H_2(\Sigma;\mathbb{Z}/2)$$
$$\cong \mathbb{Z}/2$$

since Σ is a def retract of E.

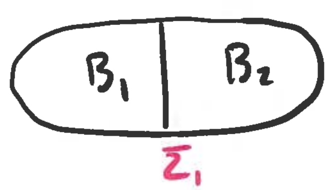
∎

Lemma ⓓ: Suppose that $\Sigma\subset Y$ is normal. Then the number of components of $Y\setminus\setminus\Sigma$ that are not twisted I-bundles is at least
$$|\Sigma| - \dim H_1(Y;\mathbb{Z}/2)+1$$
$\uparrow$ # comps of Σ.

Proof Let $\Sigma_1,\dots,\Sigma_n$ denote the components of Σ.
$B_1,\dots,B_k$ the components of $Y\setminus\setminus\Sigma$ that are twisted I-bundles.

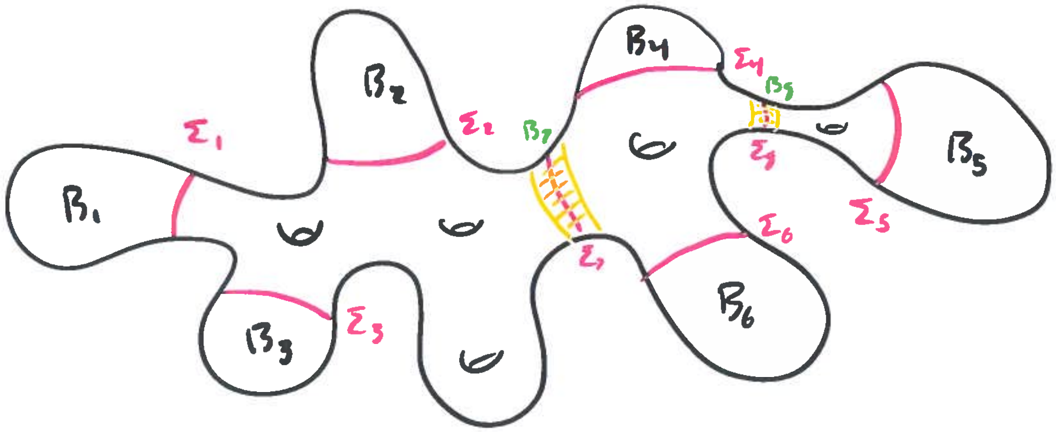
Then we can reorder the Σ_is so that
$$\partial B_i=\Sigma_i\ ,\quad 1\le i\le k$$

Note that the boundaries of the B_i must be distinct, with one exception: $n=1$, $k=2$.



In this case,
$$\dim H_1(Y;\mathbb{Z}/2)=2$$
and
$$2\ge 1-2+1=0$$

④ In every other case, we get the following picture.



In this case, $n \geq k$. Let

$$B_j = \text{nbhd}(\Sigma_j), \quad k < j \leq n$$

and

$$B = \bigcup_1^n B_i$$

Observe that the components of $\overline{Y \setminus B}$ are precisely those of $Y \setminus\!\setminus \Sigma$ that are not twisted I-bundles.

Consider the LES (w/ $\mathbb{Z}/2$ coefs) for the pair $(Y, \overline{Y\setminus B})$

$$0 \to \text{Im}(\phi) \to \quad (\dim \text{Im}(\phi) \leq m)$$

$$\begin{array}{ccccccccc} H_1(Y) & \xrightarrow{\phi} & H_1(Y, \overline{Y\setminus B}) & \longrightarrow & H_0(\overline{Y\setminus B}) & \longrightarrow & H_0(Y) & \longrightarrow & 0 \\ \cong & & \| \text{ excision} & & \cong & & \cong & & \\ (\mathbb{Z}/2)^m & & H_1(B, \partial B) & & (\mathbb{Z}/2)^p & & \mathbb{Z}/2 & & \\ & & \| & & & & & & \\ & & H_1(\coprod B_i, \coprod \partial B_i) & & & & & & \\ & & \cong \text{ (bunch of I-bundles)} & & & & & & \\ & & (\mathbb{Z}/2)^n & & & & & & \end{array}$$

So, $\dim(\text{Im}(\phi)) - n + p - 1 = 0$) χ of a LES.

$\Rightarrow$ $m \geq \dim(\text{Im}(\phi)) = n - p + 1$

$\Rightarrow$ (# non-twisted I-bundle comp'ts) $p \geq n - m + 1$

n: $|\Sigma|$, m: $\dim H_1(Y;\mathbb{Z}/2)$

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<u>Proof of theorem:</u>

Lemma D $\Rightarrow$ $Y \setminus\setminus \Sigma$ has at least
$$|\Sigma| - \dim H_1(Y; \mathbb{Z}/2) + 1$$
components that are not twisted I-bundles.

Lemma A $\Rightarrow$ $Y \setminus\setminus \Sigma$ has at most $6t$ bad components.

If
$$|\Sigma| \geq 6t + \dim H_1(Y; \mathbb{Z}/2)$$
then the number of components of $Y \setminus\setminus \Sigma$ which are not twisted I-bundles is at least
$$6t + 1$$

Not all of these components can be bad. Thus $\exists$ at least one good non twisted I-bundle components.

$\Rightarrow$ $Y \setminus\setminus \Sigma$ contains at least one product, meaning a pair of parallel surfaces $\blacksquare$

Now, let's work on proving the PDT. First, a def^{n}

Suppose that we express Y as
$$Y \cong Y_1 \# \ldots \# Y_{n+1}$$
where no $Y_i \cong S^3$. Then $\exists$ a collection of 2-spheres $S \subset Y$ st no component of $Y \setminus S$ is a punctured 3-sphere. (the summing spheres)

Call any such collection of 2-spheres an <u>independent</u> system of 2-spheres.

<u>Lemma E</u>: If Y contains an independent system of n 2-spheres, then it contains such a system which is normal.

6

Proof of PDT① (assuming Lemma Ⓔ)

Suppose that

$$Y \cong Y_1 \# \cdots \# Y_{n+1}, \quad Y_i \not\cong S^3 \text{ for each } i.$$

Then by Lemma Ⓔ, there is an independent system $\hat{S}$ of n 2-spheres which is normal.

By the previous theorem,

$$n < 6t + \dim H_1(Y; \mathbb{Z}/2)$$

since otherwise S would contain parallel comps.

Thus, the number of possible summands is bounded ♀

Let

$$Y \cong Y_1 \# \cdots \# Y_m$$

be a maximal such decomposition.

Then each of the Y_i must be prime

∎