①

Def^n The **Heegaard genus** of a closed 3-manifold Y is
$$g(Y) = \min\left\{ g(\Sigma) \;\middle|\; \begin{matrix}\Sigma \text{ a Heegaard} \\ \text{surface for } Y\end{matrix} \right\}$$

Examples

① $\underline{g(Y)=0}$: In this case we have that
$$Y = B^3 \cup_{S^2} B^3 \cong S^3$$
To see this, note that any homeomorphism
$$h : S^2 \longrightarrow S^2$$
extends to a homeomorphism
$$\tilde{h} : B^3 \longrightarrow B^3$$
via coning.

② $\underline{g(Y)=1}$: In this case, we have that
$$Y = (S^1 \times D^2) \cup_{T^2} (S^1 \times D^2)$$
So, such manifolds are determined by homeomorphisms
$$h : \underset{\underset{T^2}{\cong}}{(S^1 \times \partial D^2)} \longrightarrow \underset{\underset{T^2}{\cong}}{(S^1 \times \partial D^2)}$$

②

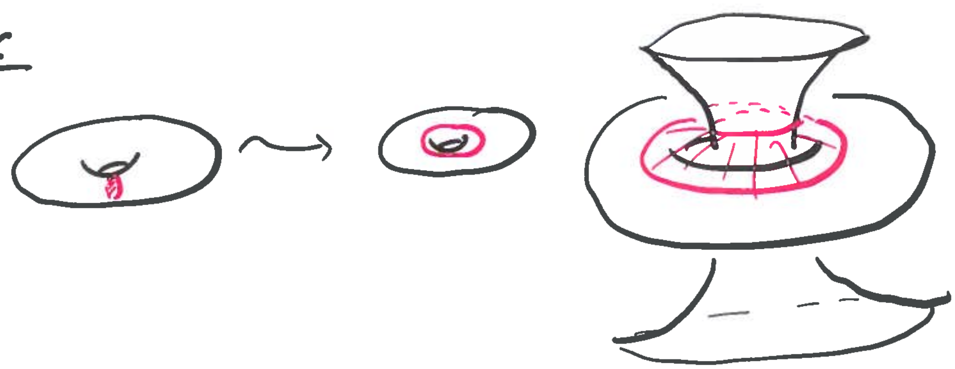
Fact: Any homeomorphism $h: T^2 \to T^2$ is isotopic to

$$h(\theta,\varphi) = \begin{pmatrix} r & p \\ s & q \end{pmatrix}\begin{pmatrix} \theta \\ \varphi \end{pmatrix} = \begin{pmatrix} r\theta + p\varphi \\ s\theta + q\varphi \end{pmatrix}$$

where $rq - sp = -1$

Note: ① When $q=0$, we get $\boxed{sp=1}$ and WLOG, we can choose $s=p=1$.

Pic



② When $p=0$, we get $\boxed{rq=-1}$, so can assume WLOG that $r=-1, q=1$.

Pic

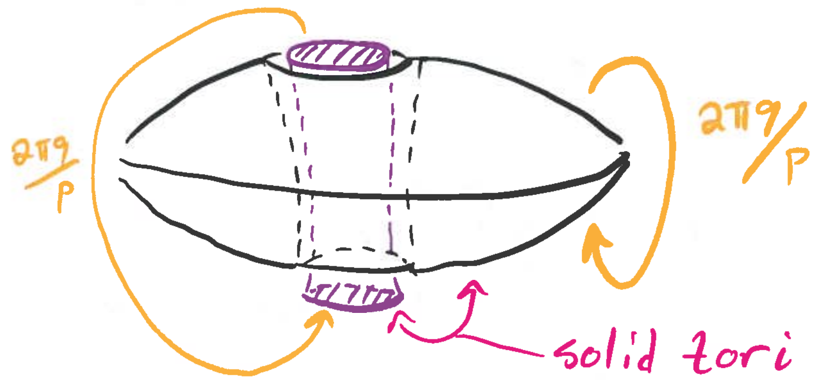
So we get an S^1's worth of S^2's

In other words, this space is $\underline{S^1 \times S^2}$ ☺

③ In general, one obtains the Lens space $L(p,q)$.

Exercise: Check that this description of $L(p,q)$ matches our previous one

Pic (from Rolfsen)



Now check that the gluing maps on the boundary of the solid tori agree.

Some easy facts to verify

① $\pi_1(L(p,q)) \cong \mathbb{Z}/p$

② $L(p,q) \cong L(p,q')$ if $q \equiv q' \mod p$.
(Just twist)

③ $L(m,q)$ is a p-fold cyclic cover of $L(mp,q)$

In particular, all Lens spaces are covered by

$$L(1,q) \cong L(1,0) = S^3$$

3 Dehn Surgery

Let $L \subset Y$ be a link and $N(L)$ a small tubular neighborhood of L.

We let

$$Y(L) := \overline{Y \setminus N(L)}$$

denote the closure of the complement of $N(L)$.

Note that if $L = K_1 \amalg \dots \amalg K_n$ is an n-component link, then

$$\partial Y(L) \cong \coprod_n T^2$$

Now, fix an (orientation reversing) homeomorphism

$$h : \partial(S^1 \times D^2) \longrightarrow \partial Y(L)$$

Let

$$Y_h(L) := Y(L) \cup_h \left(\coprod_n (S^1 \times D^2) \right)$$

Then we say that $Y_h(L)$ is the result of Dehn surgery along L.

Examples

1. $L(p,q) = S^3_{-p/q}(\text{Unknot})$
2. $PHS^3 = \begin{pmatrix}\text{Dehn surgery} \\ \text{along}\end{pmatrix}$ 

Theorem (Lickorish–Wallace)
All 3-manifolds can be obtained via surgery along a link in S^3.

To prove the L-W theorem, we need to understand a bit about the group of homeomorphisms of surfaces.

Def^n Let Σ be a surface with boundary $\partial\Sigma$. The mapping class group of the pair $(\Sigma, \partial\Sigma)$, denoted

$$MCG(\Sigma, \partial\Sigma)$$

is the group of homeomorphisms of Σ which fix $\partial\Sigma$ pointwise, up to isotopy.

To prove our theorem, we need to understand a bit about generators of this group.

⑥ Let Σ be an oriented surface, $\gamma \subset \Sigma$ a simple closed curve, and $N(\gamma) \cong \gamma \times [0,1]$ a tubular neighborhood of γ. A **positive Dehn twist** about γ is a homeomorphism

$$D_\gamma : \Sigma \to \Sigma$$

which is the identity outside $N(\gamma)$ and equal to

$$D_\gamma(\theta, t) = (\theta + 2\pi t, t)$$

on $N(\gamma)$.

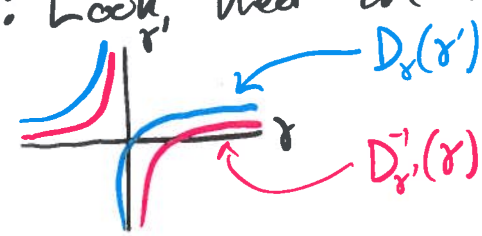
Note that D_γ has an obvious inverse which we denote D_γ^{-1}.

Let $X, X' \subset \Sigma$. If X and X' are related by a sequence of Dehn twists and isotopies, we'll write

$$X \overset{D}{\sim} X'$$

Lemma: Let $\gamma, \gamma' \subset \Sigma$ be simple closed curves which intersect in a single point, $\#(\gamma \cap \gamma') = 1$. Then $\gamma \overset{D}{\sim} \gamma'$

Proof: Look near the intersection



So, $D_\gamma(\gamma')$ and $D^{-1}_{\gamma'}(\gamma)$ are isotopic. Thus, $D_{\gamma'} \circ D_\gamma(\gamma')$ is isotopic to γ and $\gamma \overset{D}{\sim} \gamma'$. ☐