Our goal now is to prove

Theorem (Sphere Theorem)
Let Y be a 3-manifold and $f: S^2 \longrightarrow Y$ a map with $[f] \neq 0 \in \pi_2(Y)$. Then there is an embedding $e: S^2 \longrightarrow Y$ with $[e] \neq 0 \in \pi_2(Y)$.

Proof (sketch)

Let $p: \tilde{Y} \longrightarrow Y$ be the universal cover of Y and T a triangulation of Y.

We can pull T back to $\tilde{Y}$ to obtain $\tilde{T}$.

Since $\pi_1(\tilde{Y}) = 1$, $0 \neq \pi_2(\tilde{Y}) = H_2(\tilde{Y}; \mathbb{Z})$. In particular, $\exists$ a non-trivial element in $H_2(\tilde{Y})$.

Now, we know that all classes in $H_2(\tilde{Y})$ are representable by embedded, orientable surfaces.

Let $\Sigma \subset \tilde{Y}$ be such a surface with $[\Sigma] \neq 0 \in H_2(\tilde{Y})$

Since $\pi_1(\tilde{Y}) = 1$, unless $\Sigma = \amalg S^2$, we have
$$\pi_1(\Sigma) \longrightarrow \pi_1(\tilde{Y})$$
is <u>not</u> 1-1.

But, we showed previously that this implies that Σ has a compression disk.

We compress Σ along this disk to obtain a new, homologous, surface of lower genus.

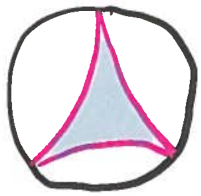
Repeating, we end up with a disjoint union of spheres, at least one of which is nonzero in $H_2(\tilde{Y}) = \pi_2(\tilde{Y})$.

Exercise: We can assume W/LOG that the sphere $\Sigma = S^2$ obtained above is normal.

Assume that Σ has been isotoped to minimize its weight
$$w(\Sigma) = \#(\Sigma \cap T^{(1)})$$
with respect to T.

Now, endow each 2-simplex of T with a hyperbolic metric via

pic



$$\frac{4}{(1-r^2)}(dx^2+dy^2)$$

Here, the edges of the 2-simplex have infinite length.

Given $\Sigma \subset Y$, define
$$\ell(\Sigma) := \sum_{\tau \text{ a 2-simplex}} \text{length of } \Sigma \cap \tau$$

Thus definition in hand, we now isotope Σ to minimize
$$(w(\Sigma), \ell(\Sigma))$$
amongst all normal reps of Σ.

It's clear that we can minimize $w(\Sigma)$. To see that it's possible to realize a min for $\ell(\Sigma)$, recall that for a fixed weight K, there are a finite number of normal spheres
$$S_1, \ldots, S_n$$
realizing $w(S_i) = K$.

Exercise: $\min\limits_{S \text{ parallel to } S_i} (\ell(S)) > 0$ and min is realized.

Thus, an absolute min is realized in either Y or $\hat{Y}$.

Let $\Sigma' = \bigcup\limits_{g \in G} g \cdot \Sigma$, where $G = \mathrm{Deck}(\hat{Y})$.

Claim: Either $\Sigma = g\Sigma$ or $\Sigma \cap g\Sigma = \emptyset$

If not, then $\exists$ a disk $D \subset \Sigma$ with
$$D \cap g\Sigma = \partial D = \gamma$$
Let D be the disk with min area amongst all possible examples in Σ or $g\Sigma$. Assume WLOG that $D \subset \Sigma$.

③ Now let • ${}_\gamma\Sigma = E \cup_\gamma E'$ dishs

• $\tilde{\tilde{\Sigma}} = E \cup D$, $\tilde{\tilde{\tilde{\Sigma}}} = E' \cup D$

Then these spheres have PL area equal to that of Σ since D is minimal.

But, we can reduce $\ell(\Sigma)$ (or $w(\Sigma)$) if our curvs intersect on a face, which must happen.

Haken Manifolds

Def A 3-manifold Y is Haken if it is compact, irreducible and contains an incompressible surface.

Def: A surface Σ is incompressible if

① Σ is properly embedded in Y

② No component of Σ is a disk for which $\partial\Sigma \subset \partial Y$ bounds a disk in ∂Y.

③ No component of Σ is an S^2 bounding a homotopy ball.

④ Σ does not have a compression disk

Recall that ④ is true $\Longleftrightarrow$ $\pi_1(\Sigma) \longrightarrow \pi_1(Y)$ is an injection

Exercise: Σ is incompressible if and only if each component of Σ is incompressible.

Proposition: If Y is Haken, then $\pi_1(Y)$ is infinite

Proof: Let Σ be an incompressible surface in Y.

Then $\Sigma \neq S^2$ since Y irreducible $\Rightarrow$ such a sphere bounds a 3-ball

Now, if $\Sigma \neq D^2$, then $\pi_1(\Sigma)$ is infinite and since $\pi_1(\Sigma) \longrightarrow \pi_1(Y)$ is an injection, so is $\pi_1(Y)$.

Finally, if $\Sigma = D^2$, then $\partial\Sigma$ is essential in ∂Y. By the Half-lives Half-dies Lemma, this implies,

$$rk(H_1(Y)) > 0$$
$$\Rightarrow \pi_1(Y) \text{ is infinite}$$

$\boxtimes$

It follows immediately that if Y is Haken, then $\widetilde{Y} \simeq \mathbb{R}^3$, and that $\pi_i(Y) = 0$ for $i \geq 2$.

Fact: There exist closed, irreducible 3-manifolds with $\pi_1(Y)$ infinite that are not Haken.

Theorem: Suppose Y is a compact 3-manifold.
1. If $H_1(Y)$ is infinite, then Y contains an incompressible, non-separating surface Σ.
2. If ∂Y has a component of genus greater than zero (which implies $H_1(Y)$ is infinite by Half lives, half dies), then the surface Σ in ① can be chosen so that $[\partial\Sigma] \neq 0$ in $H_1(\partial Y)$.

Proof: If $H_1(Y)$ is infinite, then $\exists$ an epimorphism

$$\begin{tikzcd} \pi_1(Y) \arrow[rr, two heads, "\phi"] \arrow[dr] & & \mathbb{Z} \\ & H_1(Y) \arrow[ur] \arrow[u, phantom, "\circ"] & \end{tikzcd}$$

It follows that there exists a map
$$f : Y \longrightarrow S^1$$
for which $f_* = \phi : \pi_1(Y) \longrightarrow \pi_1(S^1) \cong \mathbb{Z}$.

To see this, define $f : Y^{(1)} \longrightarrow S^1$ to be a map which looks like ϕ on the level of π_1 generators.

Since ϕ takes true relations in $\pi_1(Y)$ to true relations in $\pi_1(S^1) \cong \mathbb{Z}$, we see that f extends to a map
$$f : Y^{(2)} \longrightarrow S^1$$

(5)

Similarly, since $\pi_2(S^1)=0$, we get that f extends to

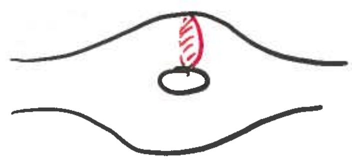
$$f: Y \longrightarrow S^1$$

With f in hand, isotope a bit to ensure that $f \pitchfork p$, a point in S^1. Let

$$\Sigma = f^{-1}(p).$$

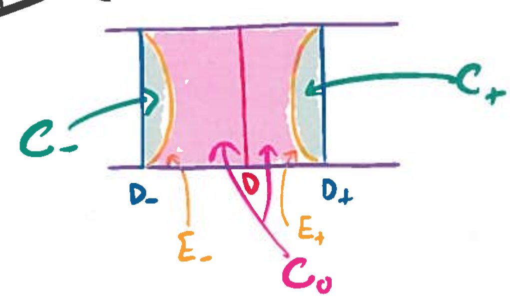
Now, either Σ is incompressible or not.

Suppose there is a compression disk D.



Let $N = D\times[-1,1]$ be a small tubular nbhd of D.

pic (cross section)



Our goal is to modify f in this neighborhood.

Define ① $\tilde{f}|_{Y\setminus N} = f|_{Y\setminus N}$

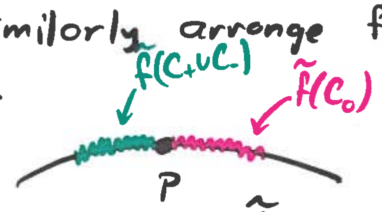
② $\tilde{f}|_{E_\pm} = p$

Note that since $[\tilde{f}(E_\pm \cup D_\pm)]=0$ in $\pi_2(S^1)$, we get that $\tilde{f}$ extends over $C_\pm$

Exercise: By possibly applying an isotopy, can arrange for $\tilde{f}^{-1}(p)\cap C_\pm = E_\pm$

We can similarly arrange for $\tilde{f}^{-1}(p)\cap C_0 = E_+ \cup E_-$

Local pic in S^1



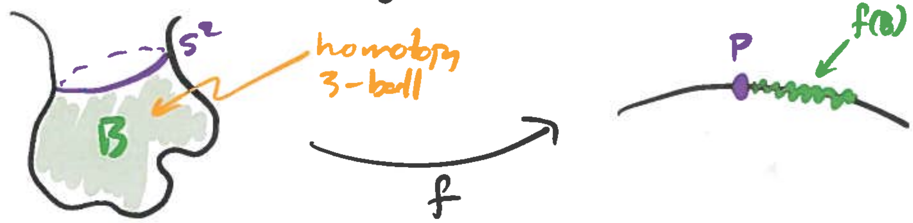
Exercise: The new map $\tilde{f}$ is isotopic to f.

We clearly see that $\tilde{f}^{-1}(p)$ is the surface obtained from Σ via surgery along D.

Iterating this process, we eventually obtain a map + inverse image with no compression disks.

⑥

Now if some component of Σ is an S^2 bounding a homotopy 3-ball



then we can isotope f so that $f(B) = p$. In turn, this gives us an isotopy of f which makes the image of $B \cup S^2$ disjoint from p.

That is, we can elliminate S^2 from $\hat{f}^{-1}(p)$.

The same argument works if Σ contains a disk D st $\partial D \subset \partial Y$ bounds a disk in ∂Y. We can remove D from $f^{-1}(p)$.

<u>Claim</u>: $f^{-1}(p) = \Sigma$ contains a non-separating component.

To see this, let $\gamma \subset Y$ be a scc which is $\pitchfork$ to Σ and which is the generator of $\pi_1(S^1)$ under f_*.

Then since $f(\gamma) \cdot p = \pm 1$, $\gamma \cdot f^{-1}(p) = \pm 1$

In particular, at least one component of $\Sigma = f^{-1}(p)$ intersects γ in an odd number of points.

<u>Exercise</u>: This component is non-seperating

Now for the second part of the proof.

② Suppose ∂Y has a component of genus > 0.

The Half-lives, half-dies lemma tells us that we can choose our epimorphism $H_1(Y) \twoheadrightarrow \mathbb{Z}$ so that

$$\begin{tikzcd} H_1(\partial Y) \arrow[r, "i_*"] \arrow[rr, bend right, "\text{non-trivial}"'] & H_1(Y) \arrow[r, two heads] & \mathbb{Z} \end{tikzcd}$$

In turn, we get Σ as in ① and also that there is a scc $\beta \subset \partial Y$ with

7

$f_*([\beta]) = k \in \pi_1(S^1) \cong \mathbb{Z}$, $k \neq 0$.

Thus, $\beta \cdot \partial\Sigma = k$ and $[\partial\Sigma] \neq 0 \in H_1(\partial Y)$

$\blacksquare$

Observe: In our proof of ①, once we obtain some non-separating surface, we can let Σ be one of minimal such genus.

Exercise: If Y is irreducible $\Rightarrow$ Σ is incompressible.

Corollary: If Y is compact, irreducible, and has $H_1(Y)$ infinite, then Y is Haken.