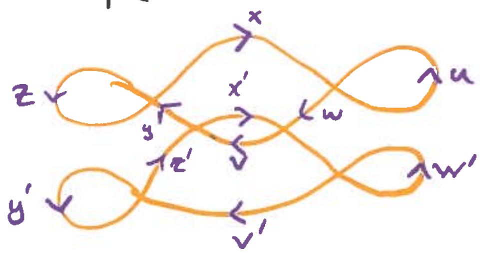
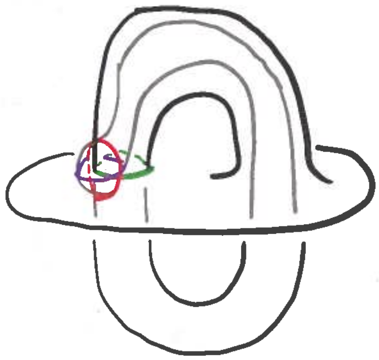
Examples

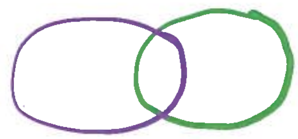
① A non-simple double-curve



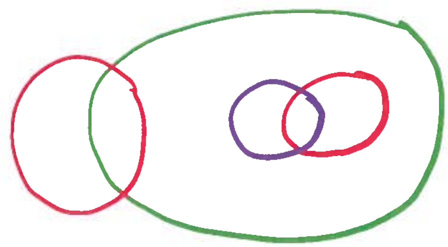
② Sometimes one can simplify intersections using covers:



Intersections on the disk



Intersections on the tube

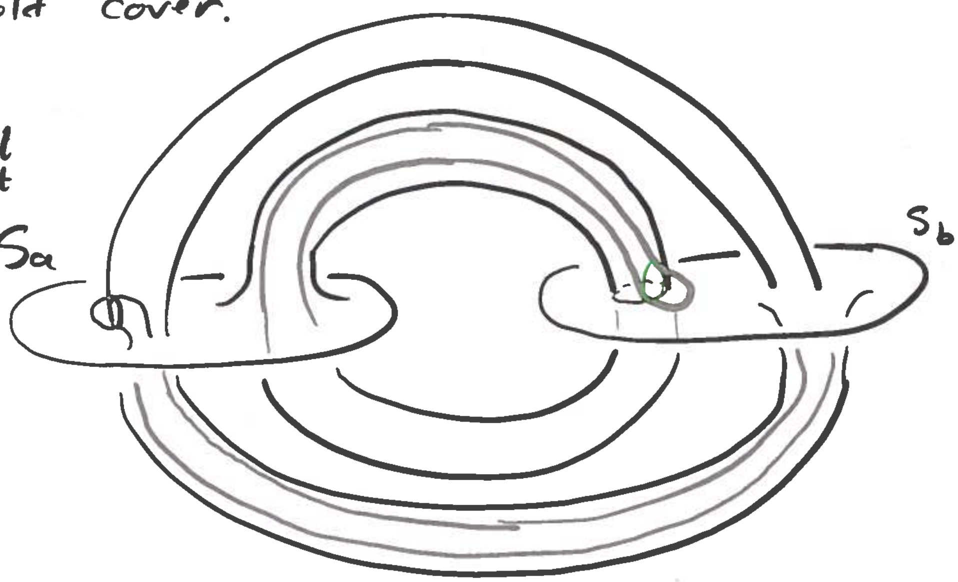


In this example, the ints are simple and could be elliminated using the previous lemma. Let's try something else.... covers:

Let N be a nbhd of this disk and look in a 2-fold cover.

pic

Notice that on S_a, get



Lemma: Let Y and f be as in the Disk thm, with $f|_{\partial D}$ an embedding. Let $\tilde{N}$ be a 2-fold cover of a regular nbhd N of $f(D)$. If $\exists$ an embedding $f_1 : D^2 \to \tilde{N}$ st $p \circ f_1(\partial D)$ is essential, then $\exists$ an embedding $e : (D^2, \partial D^2) \to (Y, \partial Y)$ st $e|_{\partial D}$ is essential.

Proof: Let $\bar{f} = p \circ f_1$. Then $\bar{f}|_{\partial D}$ is essential by assumption.

Exercise: In this setup, $\Sigma(\bar{f})$ contains only simple double-curves.

Thus, the previous lemma gives the conclusion $\boxtimes$

Lemma: Let Y and f be as in the previous lemma. If N is a regular nbhd of $f(D)$ and has no (non-trivial) 2-fold cover, then the conclusion of the disk thm holds.

Proof: No 2-fold cover means $\nexists$ an index-2 subgroup $H < \pi_1(N)$ and, in turn, no non-trivial homomorphisms

$$\pi_1(N) \longrightarrow \mathbb{Z}/2$$

(since all index-2 subgroups are normal)

Thus, there are no non-trivial maps

$$H_1(N) \longrightarrow \mathbb{Z}/2,$$

meaning that

$$H^1(N;\mathbb{Z}/2) \overset{UCT}{\cong} \overset{0}{\cancel{\mathrm{Hom}(H_1(N),\mathbb{Z}/2)}} \oplus \overset{0}{\cancel{\mathrm{Ext}(H_0(N),\mathbb{Z}/2)}}$$
$$= 0$$

Consider

$$\begin{tikzcd}
H_2(N,\partial N;\mathbb{Z}/2) \arrow[r] \arrow[d, equal, "PD"] & H_1(\partial N;\mathbb{Z}/2) \arrow[r] & H_1(N;\mathbb{Z}/2) \arrow[d, equal, "UCT"] \\
H^1(N;\mathbb{Z}/2) \arrow[d, equal] & & H^1(N;\mathbb{Z}/2) \arrow[d, equal] \\
0 & & 0
\end{tikzcd}$$

Thus, $H_1(\partial N;\mathbb{Z}/2) = 0$ and $\partial N \cong \coprod S^2$

③

In particular, $\exists\ S^2 \subset \partial N$ with $\partial D \subset S^2$. Let D' be one of the disks in S^2 bounded by ∂D.

Then we have that D' is embedded and has boundary ∂D, which is essential in ∂Y.

☑

So, if N is a regular nbhd of $f(D)$, then

(1) We're done if $\nexists$ a 2-fold cover

(2) Done if the theorem is true in a 2-fold cover

Thus, we construct a tower

$$\begin{tikzcd}
 & \vdots & \vdots \\
 & N_2 \arrow[r, phantom, "\subset"] & Y_2 \arrow[dl, "2:1"] \\
 & N_1 \arrow[r, phantom, "\subset"] \arrow[d, "2:1"'] & Y_1 \\
D^2 \arrow[r, "f=f_0"'] \arrow[ur, "f_1"] \arrow[uur, "f_2"] \arrow[uuur] & N_0 \arrow[r, phantom, "\subset"] \arrow[d, phantom, "\shortparallel"] & Y \\
 & N &
\end{tikzcd}$$

Where f_i is a lift of f_{i-1} to the cover Y_i, and N_i is a reg. nbhd of $f_i(D)$.

We call this a

Lemma: Let f, Y be as in the Disk Theorem with $f|_{\partial D}$ an embedding. Then $\exists$ a finite tower st N_n has no 2-fold cover.

Proof Consider the ith step

$$\begin{tikzcd}
 & N_i \arrow[r, phantom, "\subset"] \arrow[d, "p_i"', "2:1"] & Y_i \\
D^2 \arrow[ur, "f_i"] \arrow[r, "f_{i-1}"'] & N_{i-1} &
\end{tikzcd}$$

Then clearly $S(f_i) \subset S(f_{i-1})$

Let's suppose that $S(f_i) = S(f_{i-1})$.

Then $p_i : f_i(D) \longrightarrow f_{i-1}(D)$ induces an isomorphism on π_1

In turn, this implies,

$$\begin{tikzcd}
\pi_1(f_i(D)) \arrow[r, "i_*"] \arrow[d, "\cong"] & \pi_1(Y_i) \arrow[d, "(p_i)_*"] \\
\pi_1(f_{i-1}(D)) \arrow[r, "\cong"] & \pi_1(N_{i-1})
\end{tikzcd}$$

but this forces $(p_i)_*$ to be surjective, an impossibility for a 2:1 cover.

Thus, f_i has strictly fewer singularities than f_{i-1}. This implies that the process must terminate. ∎

At this point, we've proven the Disk Theorem when $f|_{\partial D}$ is an embedding.

Lemma: Let f and Y be as in the Disk Theorem. Then $\exists$ a map $g:(D,\partial D) \longrightarrow (Y,\partial Y)$ st $g|_{\partial D}$ is an embedding and $g(\partial D)$ is essential in ∂Y.

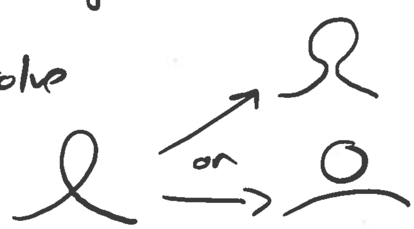
Proof: Let x be a double-point on $f(\partial D)$. Then $\exists$ a pair of preimages x' and x'' on ∂D.

In turn, we get a double-arc γ for which $x \in \partial\gamma$.

Case 1: $\partial\gamma = \{x, y\}$, where $y \in f(\partial D)$.

Pics

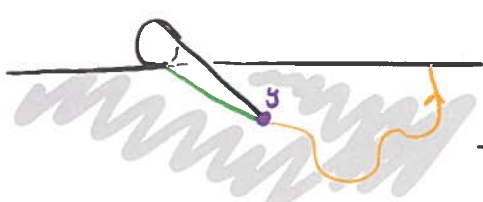
Now, resolve



One or both of these will have $\partial \neq 0$ in π_1. Both cases have fewer double-arcs.

Case 2: $\partial\gamma = \{x, y\}$, where $y \in \mathrm{int}(D)$ is a branch pt.

pic



Now, push y along an embedded arc to ∂D