Let's talk briefly about knots.

Def^{n} We say that two knots are *equivalent* if there exists an orientation preserving homeomorphism $h: S^3 \to S^3$ s.t. $h(K_1) = K_2$.

It's a fact that all OP homeos are isotopic to the identity, so K_1 is equivalent to K_2 if they are isotopic in S^3.

We say that K is *unknotted* if K is equivalent to $\bigcirc = (\mathbb{R}^2 \cap S^3) \subset \mathbb{R}^4$.

To a knot $K \subset S^3$, we can associate its fundamental group

$$K \longrightarrow \pi_1(S^3 \setminus K) =: \pi_1(K)$$

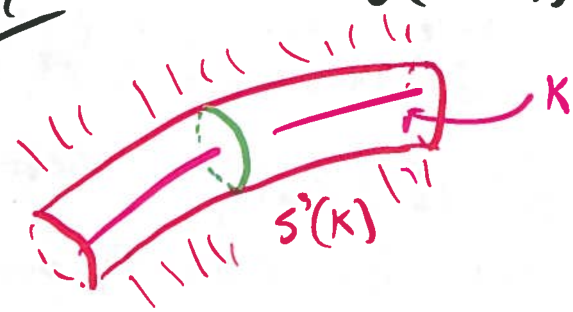
Given a knot $K \subset S^3$, we can consider its exterior

$$K \longrightarrow S^3(K) = \overline{S^3 \setminus N(K)}$$

Observe that

$$\partial(S^3(K)) \cong T^2$$

Pic



Observe that if $K = $ unknot, then $S^3(K) \cong S^1 \times D^2$ and $\pi_1(K) \cong \mathbb{Z}$.

Example

Consider $K = $ trefoil $=$ 

Exercise: $\pi_1(K) \cong \langle a, b \mid a^2 = b^3 = 1 \rangle \cong \mathbb{Z}/_2 * \mathbb{Z}/_3 \not\cong \mathbb{Z}$

Thus

② This begs the question: To what extent does $\pi_1(K)$ determine the knot K?

Theorem (Dehn, '10) Let $K \subset S^3$ be a knot. Then $\pi_1(K) \cong \mathbb{Z}$ if and only if K is the unknot.

Lemma: Let $K \subset S^3$ be a knot. Then
$$H_i(K) \cong \begin{cases} \mathbb{Z} & i = 0,1 \\ 0 & \text{else.} \end{cases}$$

proof: We give two proofs of this fact.

Quick, easy proof: We can use Alexander duality. Recall that AD says
$$\tilde{H}_i(K) \cong \tilde{H}^{n-i-1}(S^n \setminus K)$$
Since $H_i(S^1) \cong \begin{cases} \mathbb{Z} & i=0,1 \\ 0 & \text{else} \end{cases}$, $H_i(S^3 \setminus K) \cong \begin{cases} \mathbb{Z} & i=0,1 \\ 0 & \text{else}. \end{cases}$

Better proof: Let's focus on $S^3(K) = \overline{S^3 \setminus N(K)}$

pic

We have that $\partial S^3(K) = \partial N(K) \cong T^2$.

Let's apply Mayer-Vietoris to this setup.

Let $A = S^3(K)$, $B = N(K)$. Then $A \cup B = S^3$, and $A \cap B \cong T^2$.

We have that

$$\Rightarrow H_2(A) = H_2(B) = 0$$

$$0 \to \underset{\mathbb{Z}}{\underset{\cong}{H_3(S^3)}} \xrightarrow{\cong} \underset{\mathbb{Z}}{\underset{\cong}{H_2(T^2)}} \xrightarrow{0} \overbrace{H_2(A) \oplus H_2(B)} \to \underset{0}{\underset{\parallel}{H_2(S^3)}} \to$$

$$\to \underset{\mathbb{Z} \oplus \mathbb{Z}}{\underset{\cong}{H_1(T^2)}} \to H_1(A) \oplus H_1(B) \to \underset{0}{\underset{\parallel}{H_1(S^3)}}$$

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So, our attention is focused on

$$0 \longrightarrow H_1(T^2) \xrightarrow{\cong} H_1(S^3(K)) \oplus H_1(S^1 \times D^2) \longrightarrow 0$$

$$\begin{array}{ccc} H_1(T^2) & & H_1(S^1\times D^2) \\ \cong & & \cong \\ \mathbb{Z} \oplus \mathbb{Z} & & H_1(S^3(K)) \oplus \mathbb{Z} \\ (1,0) & \longmapsto & (a, 1) \\ (0,1) & \longmapsto & (b, 0) \end{array}$$

At this point, we've proven the lemma. Let's make a few observations

For this to be an isomorphism, we must have

$$b = \pm 1 \quad (\text{this is the meridian})$$

Assume WLOG, $b = 1$,

We call any curve of the form (a, b) on $T^2 = \partial N(K)$ a longitude or framing of K

Notice that,

$$(1, -a) \longmapsto 0$$

In other words, this particular curve bounds an embedded surface in $S^3(K)$.

We call this the seifert framing of K. $\blacksquare$

Proof of unknot theorem

($\Leftarrow$) If $K = \text{unknot}$, then $S^3(K) \cong S^1 \times D^2$

$$\Rightarrow \pi_1(\text{unknot}) \cong \mathbb{Z}$$

($\Rightarrow$) Let $Y = S^3(K)$ and assume $\pi_1(Y) \cong \mathbb{Z}$.

Since $Y \subset S^3$ has a single boundary component, Y is irreducible.

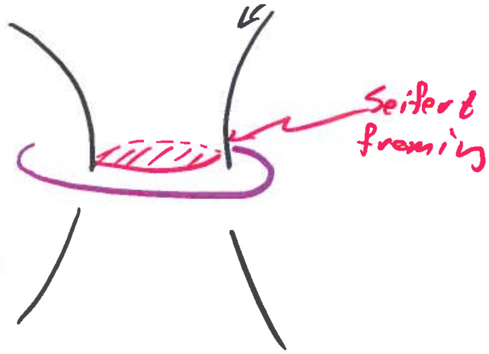
Thus, since $\pi_1(Y) \cong \mathbb{Z}$, we can apply the thm from the previous lecture to conclude that Y is a handlebody of genus $= 1$.

$S^1 \times D^2$

In particular $Y \cong S^1 \times D^2$

From the discussion in the proof of the previous lemma, we see that $\{pt\} \times \partial D^2$ represents a longitude for K and that K bounds a disk in S^3

$$\Rightarrow K = \text{Unknot}$$



∎

Unfortunately, this isn't true in general.

Example

Fact: The square knot is not isotopic to the granny (even via an OR diffeo), but

$$\pi_1(\text{Square}) \cong \pi_1(\text{Granny})$$

Theorem: If K_i is prime, then $\pi_1(K_1) \cong \pi_1(K_2)$ if and only if K_1 and K_2 are equivalent (possibly via an OR homeo).

Let's recall Dehn's Disk Theorem

Theorem: Suppose that Y is a 3-manifold and $\Sigma \subset \partial Y$ is a component of the boundary. If $\exists$ a map $f: (D^2, S^1) \to (Y, \Sigma)$ s.t. $[f|_{S^1}] \neq 0 \in \pi_1(\Sigma)$, then $\exists$ an embedding $e: (D^2, S^1) \to (Y, \Sigma)$ s.t. $[e|_{S^1}] \neq 0 \in \pi_1(\Sigma)$.

Def^n Let Σ be an embedded surface in Y. We call Σ compressible if $\exists$ a disk $D \subset Y$ s.t.

- $D \cap \Sigma = \partial D$
- ∂D is essential in Σ.

We call a closed Σ incompressible if $\Sigma \not\cong S^2$ and Σ not compressible.

Theorem: Let $\Sigma \subset Y$ be a connected, properly embedded surface. Then Σ is incompressible if and only if
$$\pi_1(\Sigma) \longrightarrow \pi_1(Y)$$
is injective.

Proof ($\Leftarrow$) Suppose to the contrary that Σ is compressible. Then $\exists$ an essential curve $\gamma \subset \Sigma$ s.t. γ bounds a disk in Y.

Thus $\pi_1(\Sigma) \longrightarrow \pi_1(Y)$ has a non-trivial kernel

($\Rightarrow$) Consider $Y \setminus\!\setminus \Sigma = \overline{Y \setminus N(\Sigma)}$, where $N(\Sigma)$ is a tubular nbhd of Σ
$$N(\Sigma) \cong \Sigma \times [-1,1]$$
Denote by $\Sigma_\pm$, the surfaces $\Sigma \times \{\pm 1\} \subset Y$.

Claim: $\pi_1(\Sigma) \longrightarrow \pi_1(Y)$ is 1-1 $\Longleftrightarrow$ $\pi_1(\Sigma_+) \longrightarrow \pi_1(Y \setminus\!\setminus \Sigma)$ is 1-1.

pf (of claim)

($\Rightarrow$) Suppose to the contrary that $\pi_1(\Sigma_+) \longrightarrow \pi_1(Y \setminus\!\setminus \Sigma)$ is not 1-1.

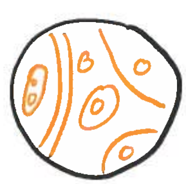
Then
$$\begin{tikzcd} \pi_1(\Sigma_+) \arrow[r] \arrow[rr, bend right, "\text{not } 1\text{-}1"'] & \pi_1(Y \setminus\!\setminus \Sigma) \arrow[r] & \pi_1(Y) \end{tikzcd}$$

Since Σ_+ is isotopic to Σ, the same is true for $\pi_1(\Sigma) \longrightarrow \pi_1(Y)$.

($\Leftarrow$) Suppose that $\pi_1(\Sigma) \longrightarrow \pi_1(Y)$ is not 1-1. Then $\exists$ a map
$$f : (D^2, S^1) \longrightarrow (Y, \Sigma)$$
s.t. $[f|_{S^1}] \neq 0 \in \pi_1(\Sigma)$.

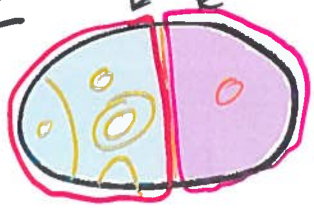
Perturb f so that $f \pitchfork \Sigma$. Then we have $f^{-1}(\Sigma)$ is a disjoint union of arcs + disks in D^2.

Pic

$f^{-1}(\Sigma) =$ 

⑥ We can assume WLOG that $f^{-1}(\Sigma)$ contains no arcs.

Pic



(one or both of these will be essential in Σ)

Let γ be an innermost SCC in $f^{-1}(\Sigma)$ and $E \subset Y$ the disk it bounds.

Case 1: $f|_\gamma$ is essential in Σ.

Then this same curve viewed on Σ_+ bounds an embedded disk in $Y \setminus\setminus \Sigma$.

Case 2: $f|_\gamma$ is inessential in Σ.

In this case, we define

$$f' : D^2 \longrightarrow Y$$

via $f'|_{D^2 \setminus E} = f|_{D^2 \setminus E}$ and $f|_E$ the disk bounded by γ in Σ.

We can homotope f to f', which has fewer intersections with Σ.

Putting everything together, if $\pi_1(\Sigma) \longrightarrow \pi_1(Y)$ is not 1-1, then neither is $\pi_1(\Sigma_+) \longrightarrow \pi_1(Y \setminus\setminus \Sigma)$.
In the latter case, $\exists$ an essential curve on Σ_+ that bounds an embedded disk in $Y \setminus\setminus \Sigma$.

Thus, Σ is compressible

$\boxtimes$