①

Theorem: Suppose that Y is a closed 3-manifold with universal cover $\tilde{Y}$.
- If $\pi_1(Y)$ is finite, then $\tilde{Y} \simeq S^3$.
- If $\pi_1(Y)$ is infinite + Y is prime then either $\tilde{Y} \simeq \mathbb{R}^3$ or $Y \cong S^1 \times S^2$

proof: • If $\pi_1(Y)$ is finite then $\tilde{Y}$ is closed.

Thus Lemma Ⓐ $\Rightarrow$ $\tilde{Y} \simeq S^3$

• Now, suppose $\pi_1(Y)$ is infinite.

Since Y is prime, either $Y \cong S^1 \times S^2$ or Y is irreducible. It suffices to show

$$Y \text{ irreducible} \Rightarrow \tilde{Y} \simeq \mathbb{R}^3.$$

Since $\pi_1(Y)$ is infinite, $\tilde{Y}$ is non-compact, $\partial \tilde{Y} = \emptyset$
Thus, by Lemma Ⓑ,

$$\tilde{Y} \simeq \mathbb{R}^3$$

$\boxtimes$

Corollary: ① If Y is a closed prime 3-manifold with $\pi_1(Y) \cong \mathbb{Z}$, then $Y \cong S^1 \times S^2$

② If Y, X are closed + prime with $\pi_1(Y) \cong \pi_1(X)$ infinite, then $Y \simeq X$.

proof: ① We begin by showing that such a manifold must have $\pi_2(Y) \neq 0$.

If not, then $\tilde{Y}$ is a non-compact 3-mfd with

$$\pi_i(\tilde{Y}) = 0 \quad \forall i \geq 1$$

and $\tilde{Y} \simeq \mathbb{R}^3$.

Now, let $f: S^1 \longrightarrow Y$ be a map representing a generator of $\pi_1(Y) \cong \mathbb{Z}$.

Since $\pi_i(S^1) = 0$ for $i \geq 2$, we have that

$$f_*: \pi_i(S^1) \longrightarrow \pi_i(Y)$$

is an isomorphism $\forall i$ and f is a homotopy equivalence by Whitehead's theorem.

In particular, $0 = H_2(Y) \overset{PD}{\cong} H^1(Y) = \mathrm{Free}(H_1(Y)) \cong \mathbb{Z}$ $\times\!\!\times$

(2)

Therefore, $\pi_2(Y) \neq 0$ and we must have $Y \cong S^1 \times S^2$

(2) Now, suppose we have $Y + X$ with $\pi_1(Y) \cong \pi_1(X)$ infinite.

If $\pi_1(Y) \cong \pi_1(X) \cong \mathbb{Z}$, then both are homeomorphic to $S^1 \times S^2$ by the above.

If $\pi_1(Y) \cong \pi_1(X) \not\cong \mathbb{Z}$, then it must be the case that $\pi_2(Y) = \pi_2(X) = 0$ and

$$\tilde{Y} \simeq \mathbb{R}^3 \simeq \tilde{X}$$

In particular,

$$\pi_i(Y) = \pi_i(X) = 0 \qquad i \geq 2$$

Thus, Y and X are both $K(\pi,1)$'s and, in turn,

$$Y \simeq X$$

$\boxtimes$

3-Manifolds with Finite π_1

Consider the Lie group $SO(4) = \left\{ \begin{matrix} \text{group of OP} \\ \text{rigid motions} \\ \text{of } \mathbb{R}^4 \end{matrix} \right\}$

$= \left\{ \begin{matrix} 4\times 4 \text{ matricies} \\ \text{with } A^TA = I \\ \text{and } \det(A) = 1 \end{matrix} \right\}$

Observe that any rigid motion of $\mathbb{R}^4$ preserves the unit sphere. Thus, $SO(4)$ acts on S^3 as well.

Let $\pi < SO(4)$ be a finite subgroup which acts freely on S^3. Then $Y = S^3/\pi$ is a closed 3-manifold with

no fixed points :o

$$\pi_1(Y) = \pi$$

and an explicit covering map

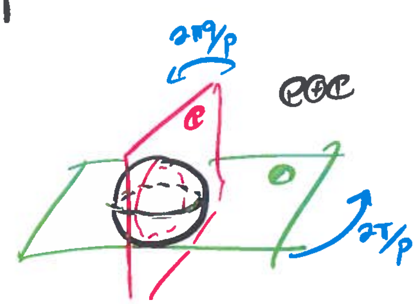
$$S^3 \xrightarrow{p} S^3/\pi = Y$$

Example

Lens Spaces: Let ξ generate $\mathbb{Z}/p$

$$\xi \cdot (z,w) = (e^{2\pi i/p} z, e^{2\pi i q/p} w)$$

pic



③

Then the action of ξ is given by the matrix

$$\xi \rightsquigarrow A = \left(\begin{array}{cc|cc} \cos(2\pi/p) & -\sin(2\pi/p) & & \\ \sin(2\pi/p) & \cos(2\pi/p) & \multicolumn{2}{c}{0} \\ \hline \multicolumn{2}{c|}{0} & \cos(2\pi q/p) & -\sin(2\pi q/p) \\ & & \sin(2\pi q/p) & \cos(2\pi q/p) \end{array}\right)$$

Elliptization Conjecture (follows from Perelman's proof of GC)

A closed 3-manifold with finite fundamental group is

$$Y = S^3/\pi$$

for some finite subgroup $\pi < SO(4)$.

Actually, a much stronger version of this theorem follows from Perelman's work

Theorem (Perelman)

Let Y and X be closed, prime 3-manifolds with $\pi_1(Y) \cong \pi_1(X)$. Then either $Y \cong X$ or both are Lens spaces

Theorem (Reidemeister '35)

Let $L(p,q)$ and $L(p,q')$ be Lens spaces

① $L(p,q)$ and $L(p,q')$ are homotopy equivalent iff

$$q' \equiv \pm a^2 q \mod p \quad \text{for some } a$$

② $L(p,q)$ and $L(p,q')$ are difeomorphic iff

$$q' \equiv \pm q^{\pm 1} \mod p$$

We might try and prove this last thm, but not quite yet.

Prime 3-manifolds with Infinite π_1

Again, we have

Theorem (Perelman)

Let Y and X be closed, prime 3-manifolds with $\pi_1(Y) \cong \pi_1(X)$. Then either $Y \cong X$ or both are Lens spaces

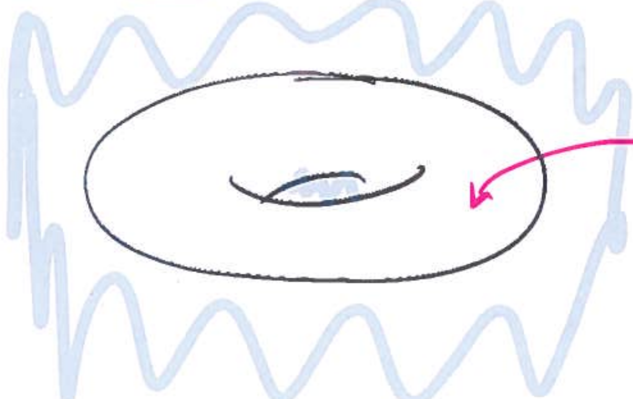
④ Parts of this theorem were known earlier, though

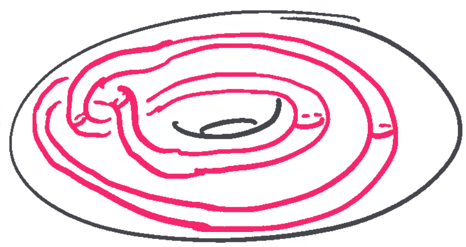
- Waldhausen showed it if Y contains an incompressible surface in '68.
- Gabai–Oertel proved it if Y has an essential lamination in 89

Theorem (Mostow Rigidity) Suppose that M + N are complete, finite volume hyperbolic manifolds and $f: M \to N$ induces an isomorphism $f_*: \pi_1(M) \to \pi_1(N)$. Then f is isotopic to a unique isometry.

If we omit the closed condition, then we have

Example (Whitehead manifold)





Now, keep going.

The result is a space W (the Whitehead manifold) with $\pi_i(W) = 0$ $\forall i$ $\Rightarrow$ W is contractible, but with W not "simply connected at ∞".
$\Rightarrow W \not\cong \mathbb{R}^3$.

Theorem: Let Y be a compact, irreducible 3-manifold with $\pi_1(Y)$ free. Then Y is a handlebody or $S^1 \times S^2$.

To prove this, we need a few lemmas

Lemma: Suppose Σ is a closed surface which is not S^2, then $\pi_1(\Sigma)$ is not free.

5

proof: Suppose to the contrary that $\pi_1(\Sigma)$ is free of rank $= n$.

Let $V = \bigvee S^1$. Then $\pi_1(V) = \ast_n \mathbb{Z}$ is free of rank $= n$.

Let $f: V \to \Sigma$ be a map which induces an isomorphism
$$f_*: \pi_1(V) \to \pi_1(\Sigma).$$
Since the universal covers of both V & Σ are contractible ($\tilde{V}$ is a "snowflake" & $\tilde{\Sigma} \cong \mathbb{R}^2$), we have that $\pi_i(V) = \pi_i(\Sigma) = 0$ $i \geq 2$.

Thus, f is a homotopy equivalence. But
$$f_*: \underset{\underset{0}{\cong}}{H_2(V)} \to \underset{\underset{\mathbb{Z}}{\cong}}{H_2(\Sigma)}$$
is not an isomorphism, a contradiction

$\times\!\!\times$ ☐

Lemma: Every subgroup of a free group is free.

proof: This is a standard exercise in topology. It follows from the fact that every cover of a graph is a graph.

☐

Lemma (Half-lives, half-dies)
Let Y be a compact, orientable 3-mfd. then the rank of the kernel of the inclusion
$$i_*: H_1(\partial Y) \to H_1(Y)$$
is half that of $H_1(\partial Y)$

Pf Since we're talking about rank, we can work over $\mathbb{Q}$. From Poincaré-Lefschetz duality, we have
$$\begin{array}{ccccc} H_2(Y,\partial Y) & \xrightarrow{\partial} & H_1(\partial Y) & \xrightarrow{i_*} & H_1(Y) \\ \cong & & \cong & & \cong \\ H^1(Y) & \xrightarrow{i^*} & H^1(\partial Y) & \xrightarrow{\delta} & H^2(Y,\partial Y) \end{array}$$
Since we're working over $\mathbb{Q}$, i^* is the Hom dual of i_*

6'

Thus,

$$\dim(\mathrm{coker}(i^*)) = \dim(\mathrm{Ker}(i_*))$$

Corollary: If Y is compact with finite $H_1(Y,\partial Y)$, then $\partial Y \cong \coprod S^2$.

proof: If ∂Y consists of something other than 2-spheres, then

$$\mathrm{rk}\, H_1(\partial Y) > 0.$$

By the previous lemma, this implies $\mathrm{rk}(H_1(Y)) > 0$ ※

proof of thm

Suppose that $\pi_1(Y)$ is free of rank $= n$. We induct on n.

Base Case: $(n=0)$. Then $\pi_1(Y) = 1$

If $\partial Y = \phi$, then $Y \simeq S^3$

If $\partial Y \neq \phi$, then $\partial Y = \coprod S^2$

Since Y is irreducible, $\partial Y = S^2$ and $Y \cong B^3$.

Inductive step $(n \geq 1)$

Since Y is irreducible, we know $\pi_2(Y) = 0$.

Since $\pi_1(Y)$ is infinite, we get that $\tilde{Y}$ is non-compact and that

$$H_i(\tilde{Y}) = 0 \implies \pi_i(Y) = 0 \quad \forall i \geq 1.$$

Let $X = \bigvee_n S^1$ and choose a map

$$f : \bigvee_n S^1 \longrightarrow Y$$

st $f_* : \pi_i(\bigvee_n S^1) \longrightarrow \pi_i(Y)$ is an isom $\forall i \geq 1$

So Whitehead $\Rightarrow$ f is a homotopy equivalence and

$$H_i(Y) = 0 \text{ for } i \geq 2.$$

In particular, since $H_3(Y) = 0$,

$$\partial Y \neq \emptyset.$$

Now, if some component of ∂Y is a 2-sphere then

$$Y \text{ irreducible } \Longrightarrow Y \cong B^3$$

Since $\pi_1(Y)$ is infinite, this can't be true.

Let $\Sigma \subset \partial Y$ be component of ∂Y.

Since $\Sigma \neq 0$, the first two lemmas above imply that

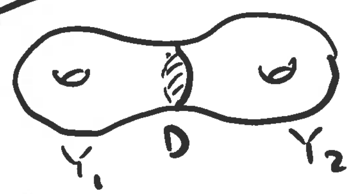
$$\pi_1(\Sigma) \longrightarrow \pi_1(Y)$$

has non-trivial kernel

Disk Thm $\Longrightarrow \exists$ a nontrivial loop on Σ that bounds a disk in Y.

<u>Case 1</u>: This disk separates Y.

<u>Pic</u>



In this case we have

$$Y \setminus\setminus D \cong Y_1 \sqcup Y_2$$

Seifert Van-Kampen $\Longrightarrow$

$$\pi_1(Y) \cong \pi_1(Y_1) * \pi_1(Y_2)$$

Since subgroups of free groups are free, $\pi_1(Y_i)$ is free of rank $= n_i$ and

$$n = n_1 + n_2$$

If either $n_i = 0$, then the base case implies that that $Y_i \cong B^3$, contradicting the fact that $[\partial D] \neq 0$ in $\pi_1(\Sigma)$.

By induction, both Y_i's are handlebodies.

<u>Case 2</u>: The disk is non-separating

Again, by SVK,

$$\pi_1(Y) \cong \pi_1(Y \setminus\setminus D) * \mathbb{Z}$$

And, again, by induction, $Y \setminus\setminus D$ and, therefore, Y are handlebodies

$\blacksquare$