①

Lemma: Let $\gamma, \gamma' \subset \Sigma$ be non-separating simple closed curves. Then $\exists$ a sequence of curves

$$\gamma = \gamma_0, \gamma_1, \dots, \gamma_n = \gamma'$$

st $\#(\gamma_i \cap \gamma_{i+1}) = 1$ for each $0 \le i \le n-1$.

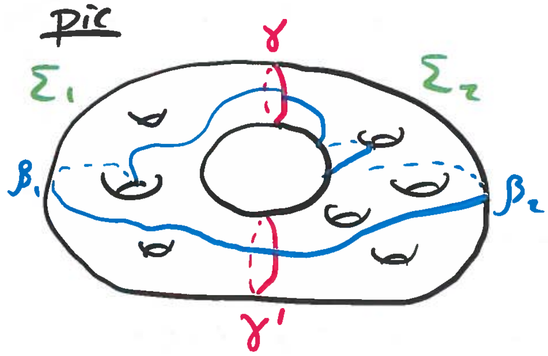
Corollary: If $\gamma, \gamma' \subset \Sigma$ are non-separating SCCs, then $\gamma \overset{D}{\sim} \gamma'$.

Proof: By induction on the number of intersections in $\gamma \cap \gamma'$.

Base Case: $\#(\gamma \cap \gamma') = 0$

Subcase 1: $\gamma \cup \gamma'$ separate Σ.

pic



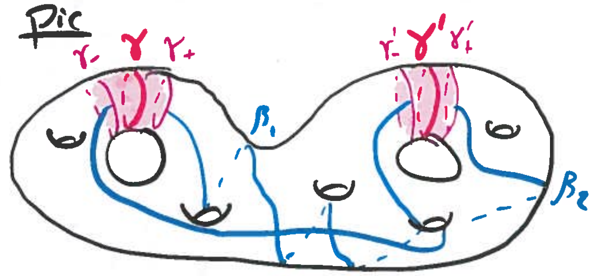
Let $\Sigma \setminus (\gamma \cup \gamma') = \Sigma_1 \cup \Sigma_2$

Choose (embedded) arcs $\beta_1 \subset \Sigma_1$, $\beta_2 \subset \Sigma_2$ so that $\mathrm{int}(\beta_i) \subset \Sigma_i$, $\partial \beta_i = \{p, p'\}$ with $p \in \gamma$, $p' \in \gamma'$.

Then $\beta_1 \cup \beta_2$ is a SCC in Σ intersecting both γ + γ' in a single point.

Subcase 2: $\gamma \cup \gamma'$ does not separate Σ.

pic



Consider

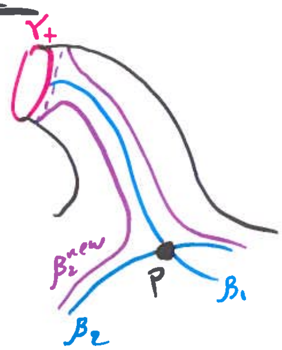
$$\partial(\Sigma \setminus (N(\gamma) \cup N(\gamma'))) = (\gamma_+ \cup \gamma_-) \cup (\gamma'_+ \cup \gamma'_-)$$

(2)

Let β_1 and β_2 be embedded arcs in $\Sigma \setminus (N(\gamma) \cup N(\gamma'))$ joinin γ_+ to γ'_- and γ'_+ to γ_-, respectively.

It could be that $\beta_1 \cap \beta_2 \neq \emptyset$. If so, then we can remove these intersections one at a time by piping.

Pic



Let p be the closest intersection point to γ_+ along β_1.

Modify β_2 as shown to obtain a new curve, β_2^{new}

Observe that β_2^{new} has fewer intersections with β_1. Thus, WLOG, we can assume

$$\beta_1 \cap \beta_2 = \emptyset.$$

If we let $\beta = \beta_1 \cup \beta_2$, then

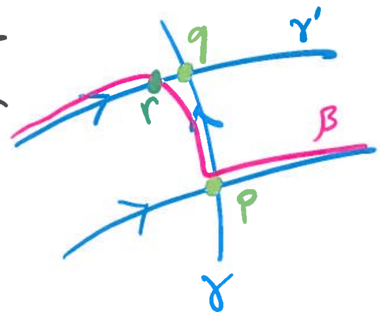
$$\#(\gamma \cap \beta) = \#(\gamma' \cap \beta) = 1.$$

Inductive Step: Assume true for $\#(\gamma \cap \gamma') < n$

Suppose $\#(\gamma \cap \gamma') = n$ and let $p + q$ be two consecutive intersections in $\gamma \cap \gamma'$ along the curve γ'.

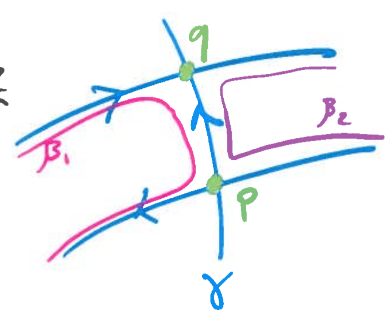
Pic

Case 1



Note that $\beta \cap \gamma' = 1$, $\beta \cap \gamma < n$

Case 2



Note that $\beta_i \cap \gamma' = 0$
$\beta_i \cap \gamma < n$

At least one of β_1, β_2 is nonseparating ▨

Def^n A complete curve system (CCS) for Σ is a maximal disjoint union of SCCs $\gamma_1, \dots, \gamma_n \subset \Sigma$ st

$$\Sigma \setminus (\gamma_1 \cup \dots \cup \gamma_n)$$

is connected.

Note The condition of being a CCS is preserved under homeomorphisms.

Theorem: Let Γ, Γ' be two CCSs for a compact surface Σ. then $\Gamma \cong \Gamma'$.

Proof

Let $\Gamma = \coprod_1^n \gamma_i$, $\Gamma' = \coprod_1^m \gamma_i'$, $m \geq n$.

The corollary above implies that $\exists$ a composition of Dehn twists h_1 st

$$h_1(\gamma_1) = \gamma_1'$$

Rinse + repeat... apply the same corollary to the pair

$$h_1(\gamma_2) \text{ and } \gamma_2'$$

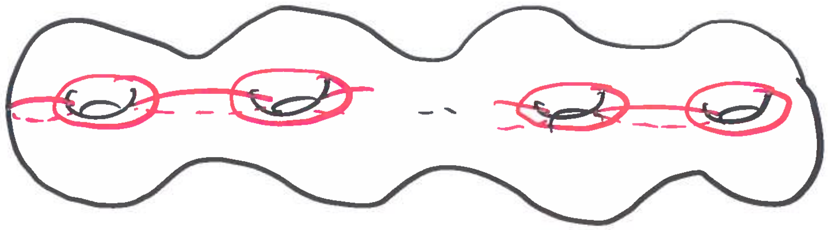
inside Σ cut along $h(\gamma_1)$. We get h_2 st

$$h_2 \circ h_1(\gamma_1) = \gamma_1'$$
$$h_2 \circ h_1(\gamma_2) = \gamma_2'$$

Now keep on going...

Fact: A tiny bit more effort shows that any homeo $h : \Sigma \to \Sigma$ with $h|_{\partial\Sigma} = Id_{\partial\Sigma}$ is isotopic to a comp of Dehn twists

④ In fact, you can use these curves



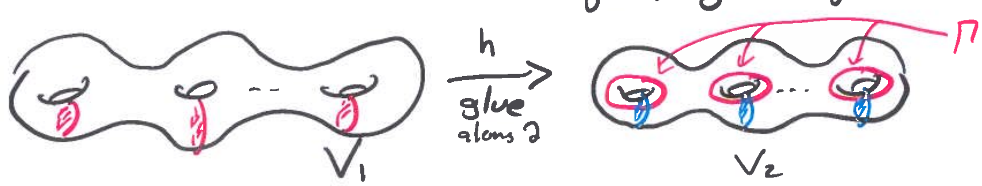
<u>Proof</u> of Lickorish-Wallace

Let Y be a closed oriented 3-manifold. Then we know that Y has a Heegaard decomp.

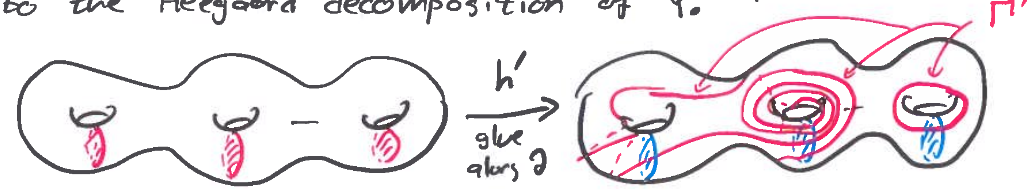
$$Y = V_1 \cup_\Sigma V_2,$$

Where V_i is a genus $= g$ handlebody.

Recall that S^3 has a "standard" genus $= g$ Heegaard decomp.



Let $h': \partial V_1 \to \partial V_2$ be the homeomorphism corresponding to the Heegaard decomposition of Y.



Observe that the Heegaard decompositions of Y + S^3 are determined by the CCSs Γ and Γ'.

From the previous theorem, we know that $\exists$ a seq of Dehn twists $D_k \circ \ldots \circ D_1 = f$ st

$$f(\Gamma) = \Gamma'$$

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Claim: There exists collections of solid tori $S_1, \dots, S_k$ and $S_1', \dots, S_k'$ contained in $\mathrm{int}(V_1)$ such that

$$f : \partial V_1 \longrightarrow \partial V_1$$

extends to

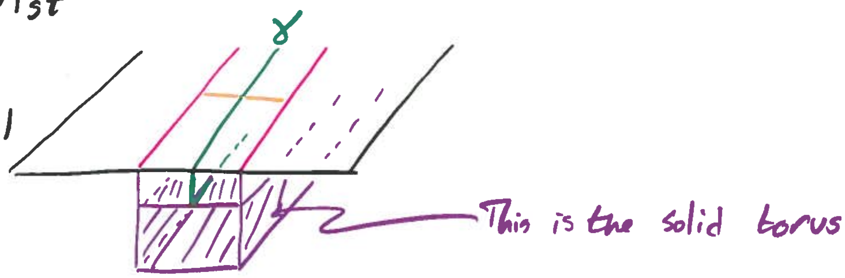
$$\tilde{f} : \overline{V_1 \setminus (\cup S_i)} \longrightarrow \overline{V_1 \setminus (\cup S_i')}.$$

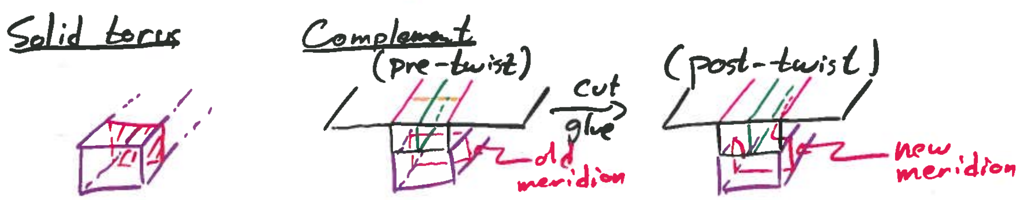
Pf of claim

Consider the picture near a single Dehn twist

Pics

(all together)





The above pictures give an explicit extension of the boundary homeomorphism to the solid torus complement

Stacking these homeomorphisms / tori on top of one another, we obtain an extension of the composition

$$D_k \circ \dots \circ D_1,$$

proving the claim $\boxed{}$

The theorem follows immediately from the claim

More precisely, by removing the relevant solid tori, one can effectuate the change in homeomorphism from h to h' along ∂V_1.

Now, simply glue the solid tori back in so that the meridians are taken to the "new" meridional curves.