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Prime Decompositions + Normal Surfaces

Recall: If Y_1 and Y_2 are connected oriented 3-manifolds, then we can form their connected sum as follows.

Let $B_i \subset \mathrm{int}(Y_i)$ be 3-balls and $\tilde{Y}_i = \overline{Y_i \setminus B_i}$. Let $h: \partial B_1 \to \partial B_2$ be an orientation reversing homeo. The connected sum of Y_1 and Y_2 is

$$Y_1 \# Y_2 = \tilde{Y}_1 \cup_h \tilde{Y}_2.$$

Facts:

1. $Y_1 \# Y_2$ is well-defined (since any two 3-balls are isotopic + any OP homeo of S^2 is isotopic to the identity).
2. The operation $\#$ is commutative, associative + has S^3 as the identity.
3. $\pi_1(Y_1 \# Y_2) = \pi_1(Y_1) * \pi_1(Y_2)$

Def: ① We call Y prime if $Y = Y_1 \# Y_2$ implies that at least one of Y_i is S^3.

② We say Y is irreducible if every embedded $S^2 \hookrightarrow Y$ bounds a 3-ball.

Proposition: If Y is irreducible, then Y is prime $\boxtimes$

Def: If $S^2 \xhookrightarrow{S} Y$ does not bound a 3-ball, then we call S an essential 2-sphere.

Theorem (Schönflies Theorem)

If $S \subset S^3$ is a smooth or PL 2-sphere, then the closures of each of the complements $S^3 \setminus S$ are 3-balls.

Note: The smooth/PL condition is essential in this theorem.

②

Example (Alexander Horned Sphere)

Observe that the curve γ is homotopically non-trivial
$\Longrightarrow$ the complement can't be B^3.

Corollary: Both S^3 and $\mathbb{R}^3$ are irreducible

Exercise: If $\Omega \subset S^3$ is a 3-manifold with connected boundary, then Ω is irreducible.

Examples

① If $K \subset S^3$ is a knot, then $\overline{S^3 \setminus N(K)}$ is irreducible.

② Since handlebodies can be realized as submanifolds of S^3, they are irreducible.

Theorem: A 3-manifold Y is prime if and only if it is either irreducible or $S^1 \times S^2$.

Proof:

Irreducible $\Rightarrow$ Prime Obvious ✓

$S^1 \times S^2 \Rightarrow$ Prime

Suppose $S^1 \times S^2 \cong Y_1 \# Y_2$

Then $\pi_1(Y_1) * \pi_1(Y_2) = \pi_1(S^1 \times S^2) \cong \mathbb{Z}$,

but this is only possible if one of $\pi_1(Y_i) = 1$.

Assume WLOG, $\pi_1(Y_1) = 1$.

Think of $\tilde{Y}_1 \hookrightarrow S^1 \times S^2$. Then we get a lift

$$\begin{tikzcd} & \mathbb{R} \times S^2 \subset \mathbb{R}^3 \arrow[d, "p"] \\ \tilde{Y}_1 \arrow[ur, "\tilde{\iota}"] \arrow[r, hook, "\iota"] & S^1 \times S^2 \end{tikzcd}$$

(it's $\mathbb{R}^3 \setminus \{0\}$)

(where $\tilde{Y}_1 = \overline{Y_1 \setminus B_3}$)

③ but $\mathbb{R}^3$ is irreducible, so $\tilde{Y}_1 \cong B^3$ ✓

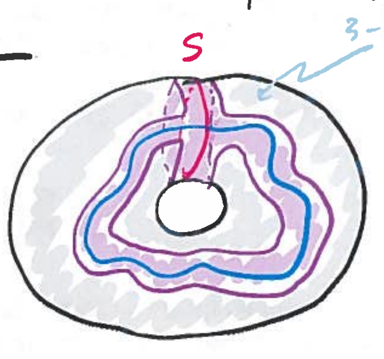
<u>Prime $\Rightarrow$ Irreducible or $S^1 \times S^2$</u>

It suffices to show that if Y is prime and not irreducible, then $Y \cong S^1 \times S^2$.

In this case, Y must contain an essential sphere, $S^2 \xhookrightarrow{S} Y$.

Since Y is prime, S must be non-separating

Pic



Let $\gamma \subset Y$ be a scc that intersects S transversally in a single point.

Let $N \cong (S \times [0,1]) \cup (\gamma \times D^2)$ be a neighborhood of $S \cup \gamma$.

Then $\partial N \cong S^2$ is a separating 2-sphere.

Since Y is prime, this 2-sphere must bound a 3-ball.

Exercise: The manifold above is $S^1 \times S^2$. $\square$

What we actually just showed is

Theorem: If Y is a 3-manifold which contains a non-separating 2-sphere, then $Y \cong (S^1 \times S^2) \# Y'$.

Theorem: Let $\tilde{Y} \xrightarrow{P} Y$ be a normal cover, then if $\tilde{Y}$ is irreducible, so is Y.

Actually, the converse is also true. If Y is irreducible, then so are all its normal covers.

Corollary: If $Y = L(p,q)$ is a Lens space (which is not $S^1 \times S^2$), then Y is irreducible.

proof: Let $S \subset Y$ be an embedded 2-sphere. Since $\pi_1(S) = 1$, $\exists$ a lift of the inclusion of S to $\tilde{Y}$.

$$\begin{tikzcd} & \tilde{Y} \arrow[d, "p"] \\ S \arrow[r, hook] \arrow[ur, dashed] & Y \end{tikzcd}$$

If $G = \mathrm{Deck}(\tilde{Y})$, then
$$p^{-1}(S) = \coprod_{g \in G} g(\tilde{S})$$

Now, since $\tilde{Y}$ is irreducible, we know that $\tilde{S} = \partial \tilde{B}$, where $\tilde{B}$ is an embedded 3-ball in $\tilde{Y}$.

If $g \neq 1$, then $g(\tilde{B}) \cap \tilde{B} = \emptyset$ since otherwise, $g(\tilde{S}) \subset \tilde{B}$, implying that $g(\tilde{B}) \subset \tilde{B}$, which cannot be true by the Brower fixed pt theorem.

Thus, $p(\tilde{B}) = B$ is an embedded 3-ball in Y with
$$\partial B = S$$

☒

Theorem:

1. (Kneser, '29) Every oriented 3-manifold can be expressed as a finite connected sum of prime 3-manifolds
2. (Milner, '62) If $Y = Y_1 \# \dots \# Y_n$ and $Y = \tilde{Y}_1 \# \dots \# \tilde{Y}_m$ are two such decompositions, then, up to reordering, $Y_1 \cong \tilde{Y}_1, \dots, Y_n \cong \tilde{Y}_m$.

The proof we will present of this result will make use of normal surface theory.

Suppose that Y is a closed 3-manifold and let
- T be a triangulation of Y
- Σ a not necessarily connected, embedded surface in Y.

By applying a small isotopy if necessary, we can assume that
$$\Sigma \pitchfork T.$$

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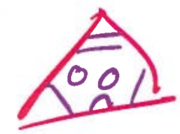
In other words

1. $\Sigma \cap T^{(0)} = \emptyset$
2. $\Sigma \cap T^{(1)} = \left\{ \begin{array}{l} \text{finite collection} \\ \text{of } \pitchfork \text{ intersection pts} \end{array} \right\}$ 
3. $\Sigma \cap \Delta = \left\{ \begin{array}{l} \text{finite collection of} \\ \text{embedded circles + arcs} \end{array} \right\}$ 

Lemma: Let $\Sigma \subset Y$ be an embedded surface + T a triangulation of Y. Then Σ can be isotoped so that for each 3-simplex $\tau \in T$, each component of $\Sigma \cap \partial\tau$ is of the form

① 0-gon ② 3-gon ③ 4-gon

Proof: Isotope the surface Σ to minimize its weight wrt $T^{(1)}$:

$$\omega(\Sigma) = \#(\Sigma \cap T^{(1)})$$

Let τ be a 3-simplex in T and C a component of $\Sigma \cap \partial\tau$.

Claim: C meets each edge of τ in at most a single point

Proof of claim: Suppose that $\#(\Sigma \cap e) > 1$.

There are two cases to consider.

Case 1: Two adjacent intersections along e have the same sign

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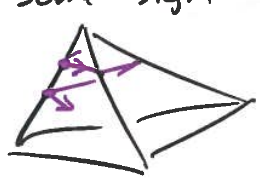

} Can't connect these up.

$\times$

Case 2: Two adjacent intersections along e have opposite sign

⑥

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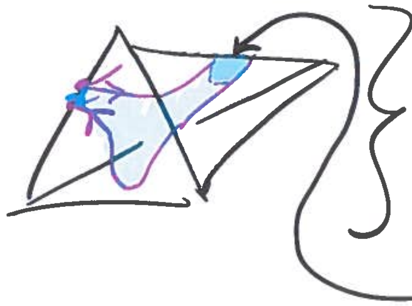
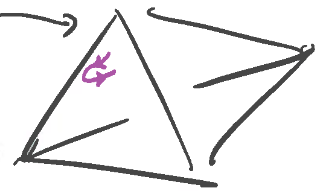


This can be isotoped to via the dish



(cbin) ∎

Thus the only possibilities are the ones listed in the Lemma

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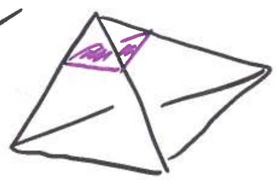
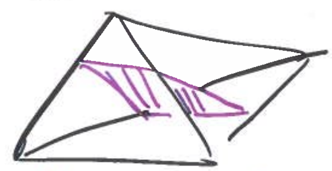
Def^n A surface Σ in a triangulated 3-manifold Y is called **normal** if for each 3-simplex $\tau \in T$, the components of $\Sigma \cap \tau$ have the following forms

Triangles



Squares



(4 of these) (3 of these)

Note that not all surfaces are normal