Transverse knots, infinite cyclic covers and Heegaard Floer homology

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47^{th} Spring Topology and Dynamics Conference

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A knot $K \subset (Y, \xi)$ is Legendrian or transverse if its tangents T_pK either lie within or transversally intersect ξ_p, respectively, for all $p \in K$.

Classical Invariants

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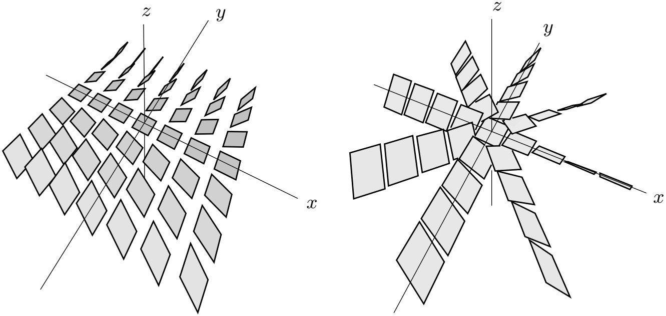
Transverse knots possess a single classical invariant, the self-linking number, denoted $\mathrm{sl}(K)$.

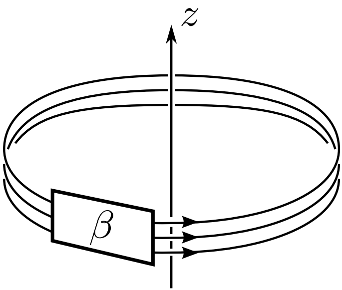
Two Contact Structures on $\mathbb{R}^3$

Below are two contact structures on $\mathbb{R}^3$. The first is $\xi_{std} = \ker(dz - y\,dx)$. The second is $\xi_{rot} = \ker(dz + r^2\,d\theta)$.



Transverse braids in $(\mathbb{R}^3, \xi_{rot})$

When working in the transverse world, it's convenient to think about a given transverse knot as being braided with respect to the z-axis in $(\mathbb{R}^3, \xi_{rot})$. This is possible by a result of Bennequin.



General Transverse braids

Definition

An *open book decomposition* for a contact 3–manifold (Y, ξ) is a pair (B, π) consisting of

- B, a transverse, fibered link
- A fibration $\pi : (Y - B) \to S^1$ by Seifert surfaces whose tangents approximate ξ.

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Definition

Suppose (B, π) is an open book compatible with the contact structure (Y, ξ). A transverse knot K in (Y, ξ) is said to be a *braid with respect to* (B, π) if K is positively transverse to the pages of (B, π).

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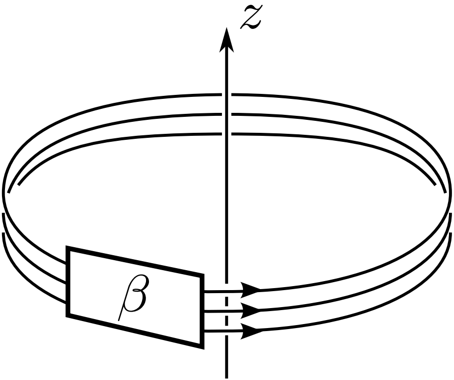

Pavelescu's Thesis

Theorem (Pavelescu)

Suppose (B, π) *is an open book compatible with* (Y, ξ)*. Then every transverse link in* (Y, ξ) *is transversely isotopic to a braid with respect to* (B, π)*.*

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Theorem (Pavelescu)

Suppose K_1 and K_2 are braids with respect to an open book (B, π) compatible with (Y, ξ). Then K_1 and K_2 are transversely isotopic if and only if there exist positive Markov stabilizations K_1^+ and K_2^+ around the binding components of (B, π) such that K_1^+ and K_2^+ are transversely isotopic with respect to (B, π).

Contact-Geometric Motivation

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There exist two transverse representatives of the $(2, 3)$*-cable of the* $(2, 3)$*-torus knot which have the same self-linking number, but which are not transversally isotopic.*

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Their proof of this theorem is a delicate argument using techniques from convex surface theory. Etnyre, LaFountain and Tosun extended this result to classify transverse representative of cables of torus knots.

Heegaard Floer homology

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In a similar spirit, if $K \subset Y$ is a null-homologous knot, then K induces a filtration on the chain complex $\widehat{\mathrm{CF}}(Y)$. The homology of this filtration $\widehat{\mathrm{HFK}}(Y, K)$ is an invariant of K called *knot Floer homology*.

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- Khandhawit and Ng used GRID to produce an infinite family of prime, transversally non-simple knot types.

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- Stipsicz and Vértesi showed that LOSS is geometrically related to a purely Legendrian invariant defined by Honda, Kazez and Matić.
- I showed that LOSS vanishes in the presence of Giroux torsion, and used this fact to show that bindings of open book decompositions must intersect each torsion layer.

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- Baldwin and Grigsby used BRAID to show that knot Floer homology solves the word problem for usual braids in $\mathbb{R}^3$.

Branched coverings

Another tool which has been fruitfully employed to study contact geometry is the theory of branched coverings.

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If $L \subset (Y, \xi)$ is a transverse knot or link and $p : \widetilde{Y} \to Y$ is a branched covering with branch-locus L, then ξ naturally extends to a contact structure on $\widetilde{Y}$.

Theorem (Giroux, Casey)

Every contact manifold is a 3-fold branched cover over some transverse link (knot) in (S^3, ξ_{std}).

Universal Knots and Links

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Theorem (Harvey-Kawamuro-Plamenevskaya, Casey)

If K is stabilized, then every branched cover over K is overtwisted.

Theorem (Casey)

If K is the Figure 8 knot, then every branched cover over K is overtwisted.

Distinguishing Transverse Knots

One could attempt to distinguish transverse knots by focusing on the contact structures obtained by taking branched covers.

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Theorem (Harvey-Kawamuro-Plamenevskaya)

Finite cyclic branched covers over transverse links L_1 and L_2 are contactomorpic if

- L_1 *and* L_2 *are related by a negative flype.*
- $L_1 = L^+$ *and* $L_2 = \overline{L}^-$, *where* $\overline{L}$ *is the Legendrian mirror of* L.

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How do we do this is a way which respects the contact geometric information associated to L?

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We begin with a transverse knot $K \subset (Y, \xi)$ which is braided about an open book decomposition (B, π).

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From the above, it's fairly straightforward to calculate the transverse invariants of lifts of K to branched covers.

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It's also not clear that this approach mirrors the the geometric problem you're trying to solve.

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Theorem

The homology $\mathrm{HFT}(Y,K)$ is an invariant of the transverse knot K. It comes equipped with a distinguished class $t(K) \in \mathrm{HFT}(Y,K)$ which is also a transverse invariant.

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- $\mathrm{HFT}(T(2, n)) = \mathbb{F}$ for all $n \geq 1$, n odd.
- $\mathrm{HFT}(U \sqcup U)$ is the free $\mathbb{F}[t, t^{-1}]$-module generated by the free group on two generators.