LEGENDRIAN CONTACT HOMOLOGY AND NONDESTABILIZABILITY

CLAYTON SHONKWILER AND DAVID SHEA VELA-VICK

We provide the first example of a Legendrian knot with nonvanishing contact homology whose Thurston–Bennequin invariant is not maximal.

1. Introduction

Since it was proposed by Etnyre [**Etn99**] and first implemented by Etnyre and Honda [**EH01**], the most common strategy for classifying Legendrian knots in a given knot type K has been to approach the problem in two steps. First, find all Legendrian representatives of K with maximal Thurston–Bennequin invariant, then attempt to show that all other Legendrian representatives of K can be destabilized to one of these maximal examples.

This method has proven quite effective, but, as observed by Etnyre and Honda [**EH05**], not all nondestabilizable Legendrian knots have maximal Thurston–Bennequin invariant. Thus, one needs a means for determining which Legendrian knots are nondestabilizable.

A candidate for identifying nondestabilizable Legendrian knots is Legendrian contact homology, which has been one of the most powerful nonclassical invariants of Legendrian knots since it was defined by Chekanov [**Che02**] and Eliashberg [**Eli98**]. This invariant, which takes the form of a differential graded algebra $(\mathcal{A}, \partial)$ and is a specialized variant of symplectic field theory [**EGH00**], vanishes for stabilized Legendrian knots and is nonvanishing for every nondestabilizable Legendrian knot for which it has been computed. All such examples to date have had maximal Thurston–Bennequin invariant, but in Theorem 1 we show that the Legendrian contact homology is nonvanishing for a certain nondestabilizable Legendrian knot with nonmaximal Thurston–Bennequin invariant.

We do this by showing that a related invariant, the characteristic algebra, is nontrivial. The *characteristic algebra* was defined by Ng [**Ng01**] as $C(L) := \mathcal{A}_F / \langle \operatorname{im} \partial \rangle$, where F is a front diagram for L, $\mathcal{A}_F$ is the free, noncommutative, unital $\mathbb{Z}/2$-algebra generated by the crossings and right cusps of F, and $\langle \operatorname{im} \partial \rangle \subset \mathcal{A}_F$ is the two-sided ideal generated by the image of

the contact homology differential. If L and L' are Legendrian isotopic, then the characteristic algebras $C(L)$ and $C(L')$ become tamely isomorphic after adding some (possibly different) number of generators to each.

Ng conjectured that the characteristic algebra of a nondestabilizable Legendrian knot is nonvanishing [**Ng01**, Conjecture 6.4.1], which would imply that the Legendrian contact homology for such knots is also nonvanishing (see Proposition 3.1). We give some evidence for Ng's conjecture by providing the first example of a Legendrian knot with nonvanishing characteristic algebra which does not have maximal Thurston–Bennequin invariant.

Theorem 1. *The contact homology and characteristic algebra of Chongchitmate and Ng's nondestabilizable Legendrian* $m(10_{161})$ *are nonvanishing.*

Remark 1.1. A similar argument to that given in the proof of Theorem 1 shows that the contact homology and characteristic algebra of Chongchitmate and Ng's nondestabilizable Legendrian $m(10_{145})$ are also nonvanishing.

Remark 1.2. There is a lift of the contact homology and characteristic algebra to $\mathbb{Z}[t,t^{-1}]$. Nonvanishing over $\mathbb{Z}/2$ implies nonvanishing in the more general $\mathbb{Z}[t,t^{-1}]$ setting.

The general situation is still far from clear, however, as we also provide some evidence against Ng's conjecture. Chongchitmate and Ng exhibited a Legendrian $m(10_{139})$ which does not have maximal Thurston–Bennequin invariant and which they conjectured, based on computational evidence, is nondestabilizable and sits atop its own peak in the tb–r mountain range. In Section 4 we prove:

Proposition 1.3. *The contact homology and characteristic algebra of Chongchitmate and Ng's Legendrian* $m(10_{139})$ *vanish identically over* $\mathbb{Z}[t,t^{-1}]$.

Assuming this knot is actually nondestabilizable, this would provide the first example of a nondestabilizable Legendrian knot with vanishing characteristic algebra or contact homology. This suggests the following:

Conjecture 1.4. *There exist nondestabilizable Legendrian knots with vanishing contact homology.*

For background information on Legendrian knots and Legendrian contact homology, we refer the reader to Etnyre's survey [**Etn05**].

2. The $m(10_{161})$

As mentioned in the introduction, Etnyre and Honda [**EH05**] presented the first example of a nondestabilizable Legendrian knot whose Thurston–Bennequin invariant is nonmaximal for its knot type. This example is a Legendrian $(2,3)$-cable of the $(2,3)$-torus knot.

Recently, Chongchitmate and Ng produced a conjectural atlas [**CN09**] for low-crossing Legendrian knots. Included in this atlas are several new examples of nondestabilizable Legendrian knots whose Thurston–Bennequin invariants are not maximal. In particular, Chongchitmate and Ng give examples of nondestabilizable Legendrian $m(10_{161})$ and $m(10_{145})$ whose Thurston–Bennequin invariants are nonmaximal (m here stands for "mirror").

For the purposes of computing the contact homology differential for a Legendrian knot, it is useful to have it presented as the plat closure of a positive braid. Using Chongchitmate and Ng's original presentation, it is not difficult to derive the plat diagram for the $m(10_{161})$ appearing in Figure 1.
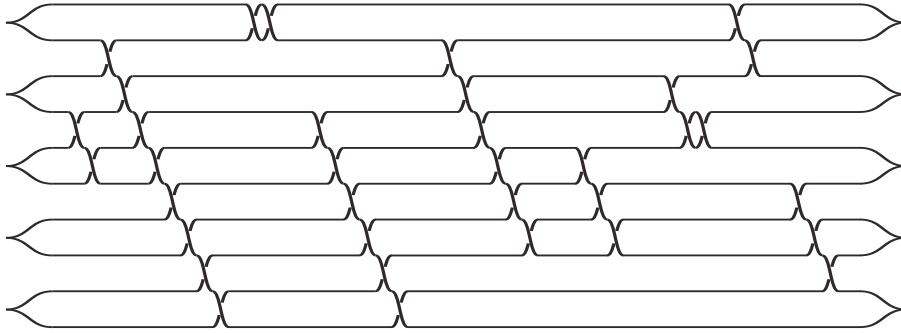

Figure 1. Chongchitmate and Ng's nondestabilizable $m(10_{161})$

The braid word defining the plat diagram in Figure 1 is:

$$4, 5, 2, 3, 4, 5, 6, 7, 8, 9, 1, 1, 4, 5, 6, 7, 8, 9, 2, 3, 4, 5, 6, 7, 5, 6, 7, 3, 4, 4, 1, 2, 6, 7, 8$$

In Figure 1 there are a total of 35 crossings and 5 right cusps. The crossings are labeled x_1 through x_{35} from left to right and the right cusps are labeled x_{36} through x_{40} from top to bottom. Therefore, for this front diagram for the $m(10_{161})$, $\mathcal{A}_{m(10_{161})}$ is equal to $\mathbb{Z}/2\langle x_1, \ldots, x_{40}\rangle$, the free unital $\mathbb{Z}/2$-algebra of rank 40 generated by $x_1, \ldots, x_{40}$. The full boundary map is given in Appendix A.

3. The Proof of Theorem 1

We begin with a straightforward observation relating (non)vanishing properties of the characteristic algebra to contact homology.

Proposition 3.1. *Let L be a Legendrian knot in the standard contact 3-sphere. If the characteristic algebra of L is nontrivial, then so is its contact homology.*

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Proof. Suppose that the contact homology

$$\mathrm{CH}(L) = \frac{\ker \partial}{\operatorname{im} \partial}$$

of L is trivial. Then, since $\partial(1) = 0$, it must be the case that $\operatorname{im} \partial$ contains the unit element 1. This implies that 1 must also be contained in the two-sided ideal $\langle \operatorname{im} \partial \rangle$ generated by the image of the boundary map inside the full algebra $\mathcal{A}_L$. Therefore, the characteristic algebra of L also vanishes, completing the proof of Proposition 3.1. □

By Proposition 3.1, Theorem 1 will follow if we can show that the characteristic algebra of the Legendrian $m(10_{161})$ depicted in Figure 1 is nontrivial.

The characteristic algebra $C(m(10_{161})) = \mathcal{A}_{m(10_{161})}/\langle \operatorname{im} \partial \rangle$ is

$$C(m(10_{161})) = \mathbb{Z}/2\langle x_1, \ldots, x_{40}\rangle / \langle \partial x_1, \ldots, \partial x_{40}\rangle.$$

From the differential we have that

$$\partial x_2 = x_1, \quad \partial x_6 = x_3, \quad \partial x_5 = x_3 x_2 + x_4,$$
$$\partial x_8 = x_7, \quad \partial x_{10} = x_9, \quad \partial x_{15} = x_{14},$$
$$\partial x_{17} = x_{16}, \quad \text{and} \quad \partial x_{26} = x_{25},$$

so, in $C(m(10_{161}))$,

(1) $$x_1 = x_3 = x_4 = x_7 = x_9 = x_{14} = x_{16} = x_{25} = 0.$$

To show that $C(m(10_{161})) \neq 0$ we will actually show that a quotient, $\overline{C} = C(m(10_{161}))/\mathfrak{I}$, is nontrivial.

Define $\mathfrak{I}$ as the two-sided ideal generated by the elements

$$x_5, x_6, x_8, x_{10}, x_{15}, x_{17}, x_{18}, x_{19}, x_{20}, x_{21}, x_{22}, x_{23}, x_{24}, x_{26}, x_{31}, x_{32}, x_{35}, x_{36},$$
$$x_{37}, x_{38}, x_{39}, x_{40}, x_{30}+1, x_{34}+1, x_{27}x_2+1, x_{11}x_2, x_{28}+x_2, x_{11}+x_{33}$$

and let

$$\overline{C} := C(m(10_{161}))/\mathfrak{I}.$$

Using (1) and the relations of $\mathfrak{I}$, the defining relations of $C(m(10_{161}))$ (i.e. the boundary maps in Appendix A) can be simplified as

(2)
$$x_2 x_{13} + x_{12} x_{11} = 1$$
$$x_{11} x_{12} + x_{27} x_{12} = 0$$
$$x_{13} x_2 = 1$$
$$x_{11}(x_{29} + 1) = 1$$
$$(x_{29} + 1)x_{11} + x_2 x_{27} = 1$$
$$x_{27} x_{12} = 1$$
$$x_{27} x_2 = 1$$
$$x_{11} x_2 = 0.$$

Therefore, $\overline{C}$ is isomorphic to $\mathbb{Z}/2\langle x_2, x_{11}, x_{12}, x_{13}, x_{27}, x_{29}\rangle$ modulo the relations in (2).

Lemma 3.2. *The algebra $\overline{C}$ is isomorphic to the algebra*

$$\mathbb{Z}/2\langle a, b, c, d\rangle/\langle ac + db = 1, ba = 0, bd = 1, ca = 1, cd = 0\rangle.$$

Proof. Define the map

$$x_{12} \mapsto a$$
$$x_{13} \mapsto b$$
$$x_{27} \mapsto c$$
$$x_{29} + 1 \mapsto d$$
$$x_2 \mapsto e$$
$$x_{11} \mapsto f.$$

Under this map, the relations in (2) become

(3) $$eb + af = 1$$
(4) $$fa + ca = 0$$
(5) $$be = 1$$
(6) $$fd = 1$$
(7) $$df + ec = 1$$
(8) $$ca = 1$$
(9) $$ce = 1$$
(10) $$fe = 0,$$

so $\overline{C}$ is isomorphic to $\mathbb{Z}/2\langle a, b, c, d, e, f\rangle$ modulo these relations.

Note that, by adding (4) to (8), the above relations imply

(11) $$fa = 1.$$

Now, we claim that the relations in (3)–(10) are equivalent to the relations

(12) $$ca = 1$$
(13) $$b + c + f = 0$$
(14) $$ba = 0$$
(15) $$a + d + e = 0$$
(16) $$cd = 0$$
(17) $$bd = 1$$
(18) $$ac + db = 1.$$

The relations (3)–(10) imply the relations (12)–(18) as follows:

- The relation (12) already appears as (8).

- Multiply (3) on the left by c and simplify using (8) and (9) to get (13).
- Multiply (13) on the right by a and simplify using (11) and (8) to get (14).
- Multiply (7) on the right by a and simplify using (11) and (8) to get (15).
- Multiply (15) on the left by c and simplify using (8) and (9) to get (16).
- Multiply (13) on the right by d and simplify using (6) and (16) to get (17).
- Finally, multiply (13) on the left by a, multiply (15) on the right by b, add the results and simplify using (3) to get (18).

On the other hand, we can derive (3)–(10) from (12)–(18) as follows:

- Multiply (13) on the left by a, add to (18) and simplify using (15) to get (3).
- Multiply (13) on the right by a and simplify using (14) to get (4).
- Multiply (15) on the left by b and simplify using (14) and (17) to get (5).
- Multiply (13) on the right by d and simplify using (16) and (17) to get (6).
- Multiply (13) on the left by d, add to (18) and simplify using (15) to get (7).
- The relation (8) appears as (12).
- Multiply (15) on the left by c and simplify using (8) and (16) to get (9).
- Finally, (13) and (15) imply that $fe = (b+c)(a+d)$; simplify using (12), (14), (16), and (17) to get (10).

Therefore, since the two collections of relations (3)–(10) and (12)–(18) are equivalent, we see that

$$\overline{C} \simeq \mathbb{Z}/2\langle a, b, c, d, e, f\rangle / \langle ca = 1, b + c + f = 0, ba = 0,$$
$$a + d + e = 0, cd = 0, bd = 1, ac + db = 1\rangle.$$

Since $e = a + d$ and $f = b + c$, we can re-write $\overline{C}$ as

$$\overline{C} \simeq \mathbb{Z}/2\langle a, b, c, d\rangle / \langle ac + db = 1, ba = 0, bd = 1, ca = 1, cd = 0\rangle,$$

completing the proof of the lemma.   □

The goal now is to show that $\overline{C}$ is nontrivial, which will imply that $C(m(10_{161}))$ is nontrivial as well.

Lemma 3.3. *The algebra*

$$\overline{C} = \mathbb{Z}/2\langle a, b, c, d\rangle / \langle ac + db = 1, ba = 0, bd = 1, ca = 1, cd = 0\rangle$$

is nontrivial.

Proof. To prove this, we define an action of $\overline{C}$ on $\mathcal{H}$, where $\mathcal{H}$ is a countably infinite-dimensional vector space over $\mathbb{Z}/2$. Provided we can show this action is nontrivial, this will imply that $\overline{C}$ is nontrivial.

As with any infinite-dimensional vector space, $\mathcal{H}$ can be written as

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2,$$

where $\mathcal{H}_1 \simeq \mathcal{H}_2 \simeq \mathcal{H}$ as $\mathbb{Z}/2$-vector spaces, so any map $\mathcal{H} \to \mathcal{H}_1 \oplus \mathcal{H}_2$ or $\mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathcal{H}$ defines an endomorphism of $\mathcal{H}$.

Fix identifications $\mathcal{H} \cong \mathcal{H}_1$ and $\mathcal{H} \cong \mathcal{H}_2$ (throughout what follows the symbol $\cong$ will refer to these fixed identifications).

Let a, b, c, d act on $\mathcal{H}$ as follows:

- Define $a : \mathcal{H} \to \mathcal{H}_1 \oplus \mathcal{H}_2$ by the diagram

$$
\mathcal{H}
\begin{array}{c}
\xrightarrow{\ \ 0\ \ } \mathcal{H}_1 \\
\oplus \\
\xrightarrow{\ \ \cong\ \ } \mathcal{H}_2.
\end{array}
$$

- Define $b : \mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathcal{H}$ by the diagram

$$
\begin{array}{c}
\mathcal{H}_1 \xrightarrow{\ \ \cong\ \ } \\
\oplus \qquad\qquad \mathcal{H}. \\
\mathcal{H}_2 \xrightarrow{\ \ 0\ \ }
\end{array}
$$

- Define $c : \mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathcal{H}$ by the diagram

$$
\begin{array}{c}
\mathcal{H}_1 \xrightarrow{\ \ 0\ \ } \\
\oplus \qquad\qquad \mathcal{H}. \\
\mathcal{H}_2 \xrightarrow{\ \ \cong\ \ }
\end{array}
$$

- Define $d : \mathcal{H} \to \mathcal{H}_1 \oplus \mathcal{H}_2$ by the diagram

$$
\mathcal{H}
\begin{array}{c}
\xrightarrow{\ \ \cong\ \ } \mathcal{H}_1 \\
\oplus \\
\xrightarrow{\ \ 0\ \ } \mathcal{H}_2.
\end{array}
$$

Extending by linearity, the defining relations of $\overline{C}$ are preserved by this action, so the above induces a well-defined action of $\overline{C}$ on $\mathcal{H}$ (alternatively, a representation of $\overline{C}$ into $\operatorname{End}(\mathcal{H})$). Since the actions of a, b, c, and d are clearly nontrivial, this is a nontrivial action, completing the proof of the lemma. $\qquad\square$

Since $\overline{C}$ is a quotient of $C(m(10_{161}))$, Lemma 3.3 implies that $C(m(10_{161}))$ is nontrivial, completing the proof of Theorem 1.

4. The Contact Homology of the $m(10_{139})$

Our goal in this section is to prove Proposition 1.3 by showing that 1 is in the image of the differential of Chongchitmate and Ng's conjecturally

nondestabilizable $m(10_{139})$. This Legendrian $m(10_{139})$ is one of two exam-
ples given by Chongchitmate and Ng with nonmaximal Thurston–Bennequin
invariants which computations suggest sit atop their own peaks in the tb–r
mountain range. That the other—a Legendrian $m(12n_{242})$—also has van-
ishing contact homology and characteristic algebra follows from a similar
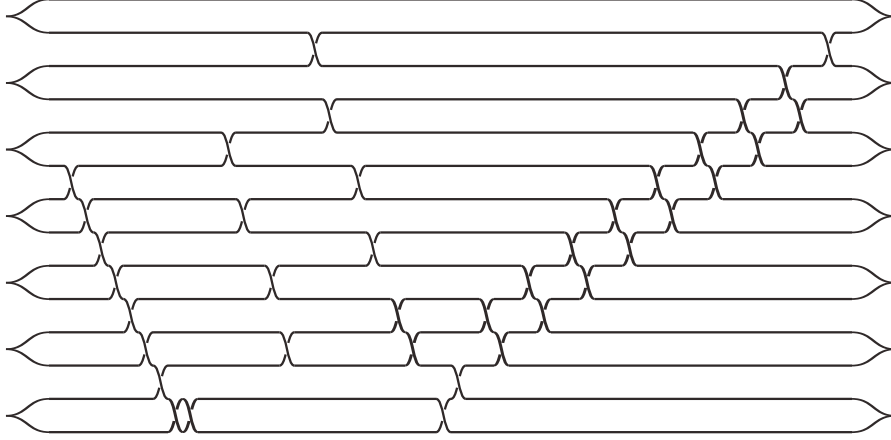argument to the one given below.

Figure 2. Chongchitmate and Ng's conjecturally nondesta-
bilizable $m(10_{139})$

The plat diagram for the $m(10_{139})$ given in Figure 2 is obtained from
Chongchitmate and Ng's presentation.

The braid word for the plat diagram in Figure 2 is:

$$6, 7, 8, 9, 10, 11, 12, 13, 13, 5, 7, 9, 11, 2, 4, 6, 8, 10, 11,$$
$$13, 12, 10, 11, 9, 10, 8, 9, 7, 8, 6, 7, 5, 6, 4, 5, 3, 4, 2$$

In order to prove that the contact homology and characteristic algebra of
the $m(10_{139})$ are trivial, it suffices to construct an element $a \in A$ such that
$\partial a = 1$. From the presentation of the differential given in Appendix B, we
see that

$$1 = \partial \left((x_2 + x_{10}) \Big(((x_{41}x_{11} + x_{14}x_{42})x_{15} + x_{41} - x_{44}) \, x_{22} + x_{24} \right.$$

$$\left. + (x_4 + x_{16})(x_{15}x_{22} + x_{19}) + x_6 + x_{43} \Big) \right).$$

Therefore, the contact homology and characteristic algebra of Chongchit-
mate and Ng's $m(10_{139})$ both vanish over $\mathbb{Z}[t, t^{-1}]$.

Acknowledgements

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Appendix A. The Differential over $\mathbb{Z}/2$ for the $m(10_{161})$

$\partial x_1 = 0$

$\partial x_2 = x_1$

$\partial x_3 = 0$

$\partial x_4 = x_3 x_1$

$\partial x_5 = x_3 x_2 + x_4$

$\partial x_6 = x_3$

$\partial x_7 = 0$

$\partial x_8 = x_7$

$\partial x_9 = 0$

$\partial x_{10} = x_9$

$\partial x_{11} = x_3$

$\partial x_{12} = 0$

$\partial x_{13} = 0$

$\partial x_{14} = 0$

$\partial x_{15} = x_{14}$

$\partial x_{16} = 0$

$\partial x_{17} = x_{16}$

$\partial x_{18} = 0$

$\partial x_{19} = x_1 + x_{12} x_4 + x_{12} x_{11} x_1$

$\partial x_{20} = x_2 x_{13} + x_{12} x_5 x_{13} + x_{12} x_{11} x_2 x_{13} + 1 + x_{12} x_6 + x_{12} x_{11} + x_{19} x_{13}$

$\partial x_{21} = x_2 x_{14} + x_{12} x_5 x_{14} + x_{12} x_{11} x_2 x_{14} + x_{12} x_7 + x_{19} x_{14}$

$\partial x_{22} = x_2 x_{15} + x_{12} x_5 x_{15} + x_{12} x_{11} x_2 x_{15} + x_{12} x_8 + x_{19} x_{15} + x_{21}$

$\partial x_{23} = x_2 x_{16} + x_{12} x_5 x_{16} + x_{12} x_{11} x_2 x_{16} + x_{12} x_9 + x_{19} x_{16}$

$\partial x_{24} = x_2 x_{17} + x_{12} x_5 x_{17} + x_{12} x_{11} x_2 x_{17} + x_{12} x_{10} + x_{19} x_{17} + x_{23}$

$\partial x_{25} = 0$

$\partial x_{26} = x_{25}$

$\partial x_{27} = 0$

$\partial x_{28} = 0$

$\partial x_{29} = x_{28} x_{25}$

$\partial x_{30} = 0$

$\partial x_{31} = x_4 + x_{11} x_1$

$\partial x_{32} = x_5 x_{13} x_{28} + x_{11} x_2 x_{13} x_{28} + x_6 x_{28} + x_{11} x_{28} + x_5 x_{14} + x_{11} x_2 x_{14} + x_7 + x_{31} x_{13} x_{28} + x_{31} x_{14}$

$\partial x_{33} = 0$

$\partial x_{34} = 0$

$\partial x_{35} = x_{33} x_2 x_{18} + x_{33} x_{12} x_5 x_{18} + x_{33} x_{12} x_{11} x_2 x_{18} + x_{33} x_{12} + x_{33} x_{19} x_{18} + x_{34} x_{27} x_2 x_{18} + x_{34} x_{27} x_{12} x_5 x_{18} + x_{34} x_{27} x_{12} x_{11} x_2 x_{18} + x_{34} x_{27} x_{12} + x_{34} x_{27} x_{19} x_{18}$

$\partial x_{36} = x_{13} x_{28} + x_{14} + 1$

$\partial x_{37} = x_5 x_{13} x_{29} x_{30} + x_{11} x_2 x_{13} x_{29} x_{30} + x_6 x_{29} x_{30} + x_{11} x_{29} x_{30} + x_5 x_{15} x_{25} x_{30} + x_{11} x_2 x_{15} x_{25} x_{30} + x_8 x_{25} x_{30} + x_5 x_{16} x_{30} + x_{11} x_2 x_{16} x_{30} + x_9 x_{30} + x_5 x_{13} +$

$$x_{11}x_2x_{13}+x_6+x_{11}+x_{31}x_{13}x_{29}x_{30}+x_{31}x_{15}x_{25}x_{30}+x_{31}x_{16}x_{30}+x_{31}x_{13}+$$
$$x_{32}x_{25}x_{30}+1$$
$$\partial x_{38} = x_{33} + x_{30}x_{28}x_{26}x_{33} + x_{30}x_{29}x_{33} + x_{30}x_{28}x_{27} + 1$$
$$\partial x_{39} = x_{27}x_2x_{18} + x_{27}x_{12}x_5x_{18} + x_{27}x_{12}x_{11}x_2x_{18} + x_{27}x_{12} + x_{27}x_{19}x_{18} + 1$$
$$\partial x_{40} = x_{33}x_2 + x_{33}x_{12}x_5 + x_{33}x_{12}x_{11}x_2 + x_{33}x_{19} + x_{34}x_{27}x_2 + x_{34}x_{27}x_{12}x_5 +$$
$$x_{34}x_{27}x_{12}x_{11}x_2 + x_{34}x_{27}x_{19} + 1$$

Appendix B. The Differential over $\mathbb{Z}[t, t^{-1}]$ for the $m(10_{139})$

$$\partial x_1 = 0$$
$$\partial x_2 = -x_1$$
$$\partial x_3 = 0$$
$$\partial x_4 = -x_3$$
$$\partial x_5 = 0$$
$$\partial x_6 = -x_5$$
$$\partial x_7 = 0$$
$$\partial x_8 = -x_7$$
$$\partial x_9 = 0$$
$$\partial x_{10} = x_1$$
$$\partial x_{11} = 0$$
$$\partial x_{12} = 0$$
$$\partial x_{13} = 0$$
$$\partial x_{14} = 0$$
$$\partial x_{15} = 0$$
$$\partial x_{16} = x_2x_{11} + x_{10}x_{11} + x_3$$
$$\partial x_{17} = x_{11}x_{12}$$
$$\partial x_{18} = x_{12}x_{13}$$
$$\partial x_{19} = x_{12}$$
$$\partial x_{20} = 0$$
$$\partial x_{21} = x_{18}x_9x_{20} - x_{19}x_{13}x_9x_{20} + x_{18} - x_{19}x_{13}$$
$$\partial x_{22} = 0$$
$$\partial x_{23} = x_9x_{20} + 1 + x_{22}x_{13}x_9x_{20} + x_{22}x_{13}$$
$$\partial x_{24} = x_{11}x_{18}x_{22} + x_{17}x_{13}x_{22} + x_{11}x_{19} + x_{17}$$
$$\partial x_{25} = x_{11}x_{18}x_{23} + x_{17}x_{13}x_{23} + x_{11}x_{21} - x_{24}x_{13}x_9x_{20} - x_{24}x_{13}$$
$$\partial x_{26} = x_{13}x_{22} + 1$$
$$\partial x_{27} = x_{13}x_{23} + x_{26}x_{13}x_9x_{20} + x_{26}x_{13}$$
$$\partial x_{28} = x_2x_{17}x_{26} + x_{10}x_{17}x_{26} + x_4x_{12}x_{26} + x_5x_{26} + x_{16}x_{12}x_{26} + x_2x_{24} + x_{10}x_{24} +$$
$$x_4x_{18}x_{22} + x_{16}x_{18}x_{22} + x_6x_{13}x_{22} + x_7x_{22} + x_4x_{19} + x_{16}x_{19} + x_6$$
$$\partial x_{29} = x_2x_{17}x_{27} + x_{10}x_{17}x_{27} + x_4x_{12}x_{27} + x_5x_{27} + x_{16}x_{12}x_{27} + x_2x_{25} + x_{10}x_{25} +$$
$$x_4x_{18}x_{23} + x_{16}x_{18}x_{23} + x_6x_{13}x_{23} + x_7x_{23} + x_4x_{21} + x_{16}x_{21} + x_8x_9x_{20} +$$
$$x_{20} + x_8 - x_{28}x_{13}x_9x_{20} - x_{28}x_{13}$$
$$\partial x_{30} = x_{12}x_{26} + x_{18}x_{22} + x_{19}$$
$$\partial x_{31} = x_{12}x_{27} + x_{18}x_{23} + x_{21} + x_{30}x_{13}x_9x_{20} + x_{30}x_{13}$$

$\partial x_{32} = x_{15}x_{11}x_{30} + x_{15}x_{17}x_{26} + x_{15}x_{24}$
$\partial x_{33} = x_{15}x_{11}x_{31} + x_{15}x_{17}x_{27} + x_{15}x_{25} - x_{32}x_{13}x_9x_{20} - x_{32}x_{13}$
$\partial x_{34} = x_{11}x_{30} + x_{17}x_{26} + x_{24}$
$\partial x_{35} = x_{11}x_{31} + x_{17}x_{27} + x_{25} + x_{34}x_{13}x_9x_{20} + x_{34}x_{13}$
$\partial x_{36} = x_{14}x_{15}x_{34} + x_{14}x_{32}$
$\partial x_{37} = x_{14}x_{15}x_{35} + x_{14}x_{33} - x_{36}x_{13}x_9x_{20} - x_{36}x_{13}$
$\partial x_{38} = x_{15}x_{34} + x_{32}$
$\partial x_{39} = x_{14}x_{38} + x_{36} + 1$
$\partial x_{40} = x_{15}x_{35} + x_{33} + x_{38}x_{13}x_9x_{20} + x_{38}x_{13} + 1$
$\partial x_{41} = x_{14}x_{15} + 1$
$\partial x_{42} = x_{15}x_{11} + 1$
$\partial x_{43} = x_2x_{17} + x_{10}x_{17} + x_4x_{12} + x_5 + x_{16}x_{12} + 1$
$\partial x_{44} = x_{11}x_{18} + x_{17}x_{13} + 1$
$\partial x_{45} = x_{18}x_9 - x_{19}x_{13}x_9 + t^{-1}$

DEPARTMENT OF MATHEMATICS
HAVERFORD COLLEGE
E-mail address: cshonkwi@haverford.edu
http://www.haverford.edu/math/cshonkwi

DEPARTMENT OF MATHEMATICS
COLUMBIA UNIVERSITY
E-mail address: shea@math.columbia.edu
http://www.math.columbia.edu/~shea