Contact structures and knot Floer homology

David Shea Vela-Vick

Joint with John B. Etnyre and Rumen Zarev

Department of Mathematics
Louisiana State University

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Motivation

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Motivating the search for such invariants was a desire to define transverse knot invariants and better understand the tight contact structures supported by these spaces.

Knot invariants

The genesis of these ideas came out of an attempt to better understand transverse knot invariants defined by Honda, Kazez and Matić, and Lisca, Ozsváth, Stipsicz and Szabo. To a knot $K \subset Y$, we associate a collection of groups and maps:

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Definition

We define the sutured limit homology of the pair (Y, K) to be

$$\underrightarrow{\mathrm{SFH}}(Y, K) := \varinjlim(\mathrm{SFH}(Y(K), \Gamma_i).$$

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Theorem (with Etnyre and Zarev)

There is a (multi-)graded $\mathbb{F}[U]$-module isomorphism

$$\underrightarrow{\mathrm{SFH}}(Y,K) \cong \mathrm{HFK}^-(Y,K).$$

Honda, Kazez and Matić's Legendrian invariant

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Definition (Honda-Kazez-Matić)

Let $L \subset (Y, \xi)$ be a Legendrian knot and let $N(L)$ be an open standard neighborhood of L. Then,

$$\mathrm{EH}(L) := \mathrm{EH}(Y \backslash N(L), \xi|_{Y \backslash N(L)}) \in SFH(-(Y \backslash N(L)), \Gamma_L).$$

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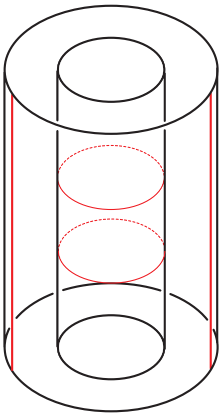
Note: $\mathrm{EH}(L)$ changes under negative stabilization and does not, therefore, define a transverse invariant.

The Stipsicz-Vertesi correspondence



- Stipsicz and Vertesi defined

  $$\Phi : SFH(-Y(L), \Gamma_L) \to \widehat{\mathrm{HFK}}(-Y, L),$$

  identifying $\mathrm{EH}(L)$ and $\widehat{\mathfrak{L}}(L)$ by attaching a basic slice to the boundary of $Y(L)$.

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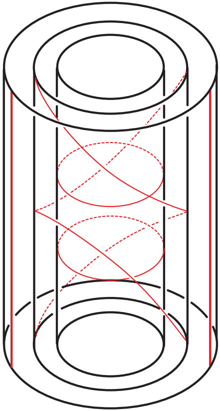

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  identifying $\mathrm{EH}(L)$ and $\widehat{\mathfrak{L}}(L)$ by attaching a basic slice to the boundary of $Y(L)$.
- There are two possible choices for the “sign” of this basic slice. They choose the one which is compatible with negative stabilization.

Constructing the limit

From this perspective, it's clear why the LOSS hat-invariant vanishes for a positively stabilized Legendrian knot.

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$$(Y(L), \xi_L), (Y(L'), \xi_{L'}), (Y(L''), \xi_{L''}), \dots$$

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Each term in the sequence is obtained from its predecessor by a basic slice attachment. Honda, Kazez and Matić's gluing theorem gives us

$$\mathrm{SFH}(-Y(L), \Gamma_L) \xrightarrow{\phi_1} \mathrm{SFH}(-Y(L'), \Gamma_{L''}) \longrightarrow \dots$$

The limit invariant

This is (up to orientation) the sequence

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described earlier. Its directed limit is $\underrightarrow{\mathrm{SFH}}(Y, L)$.

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The U-action is defined by inserting a positively signed basic slice somewhere in the above sequence.

Although its definition was contact geometric in nature, the group $\underrightarrow{\mathrm{SFH}}(Y,L)$ depends only on the topological knot type of L.

A Legendrian/transverse invariant

Since the maps ϕ_i respect the contact invariants, we get a Legendrain/transverse invariant

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- $\underrightarrow{\mathrm{EH}}(L^-) = \underrightarrow{\mathrm{EH}}(L)$
- $\underrightarrow{\mathrm{EH}}(L^+) = U \cdot \underrightarrow{\mathrm{EH}}(L)$
- If $\mathrm{EH}(Y, \xi) \neq 0$, then $\underrightarrow{\mathrm{EH}}(L) \neq 0$ for Legendrian $L \subset (Y, \xi)$.

Correspondence Theorem

Theorem (with Etnyre and Zarev)

Let $K \subset Y$ be a nullhomologous knot, then there is a (multi-)graded $\mathbb{F}[U]$-module isomorphism

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Idea of proof: Use bordered sutured Floer homology...

Sketch of proof

After isolating a collar neighborhood of the boundary, we compute a sequence of modules

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and pairs of maps ϕ_i and ψ_i between them.

The limit of these modules along the ϕ_i is a familiar module from BFH which computes the minus flavor of knot Floer homology.

The U-action is obtained by inserting the map ψ_j somewhere in the sequence and agrees with the U-action in knot Floer homology.

A generalization

Similar ideas allow one to define invariants of 3-manifolds with $T^2 \times [0, \infty)$-ends, together with a "slope" at infinity.

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Contact structures on such 3-manifolds have well-defined slopes at infinity, and we can define contact invariants which take values the appropriate limit group

A further generalization

One can further generalize these ideas to produce homological invariants of transverse knots in contact 3-manifolds.

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In Joint work with Lidman and Sarkar, we show that, by focusing on the infinite cyclic cover, the transverse invariant in knot Floer homology can be lifted to a full homology theory.

Theorem (with Lidman and Sarkar)

There exists a homology theory $\mathrm{HFT}(Y, K)$ *associated to the infinite cyclic cover of a transverse knot* K *which is an invariant of the transverse knot type of* K*. It comes equipped with a distinguished class* $t(K) \in \mathrm{HFT}(Y, K)$ *which is also a transverse invariant.*