Refining the Ozsváth-Szabó contact invariant

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Motivation

Broadly speaking, out goal is to define an effective and computable invariants of contact structures.

Motivating the search for such invariants was a desire to better understand the contact-geometric information contained in Heegaard Floer theory.

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To a knot K inside a 3-manifold Y one obtains a filtration of this complex

$$K \subset Y \longrightarrow \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n = \widehat{CF}(Y) \longrightarrow \widehat{HFK}(Y, K).$$

A contact invariant

Roughly a decade ago, Ozsváth and Szabó identified an invariant of contact structures taking values in $\widehat{HF}(-Y)$. If (Y, ξ) is a closed contact manifold, their invariant is denoted

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(c) If (Y', ξ') is obtained from (Y, ξ) by $(+1)$-contact surgery, then $c(Y, \xi)$ is sent to $c(Y, \xi)$ under the associated 2-handle cobordism map.

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(c) Ghiggini, Honda and van Horn-Morris showed that positive (2π)-Giroux torsion implies $c = 0$.

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The dream: To a contact 3–manifold (Y, ξ) define a filtration.

$$(Y, \xi) \longrightarrow \mathfrak{F}_0 \subset \mathfrak{F}_1 \subset \cdots \subset \mathfrak{F}_n = \widehat{CF}(Y) \longrightarrow \widehat{HFC}(Y, \xi).$$

so that the filtered homotopy type of the result is a contact invariant.

A definition review

Let $\mathfrak{ob} = (B, \pi)$ be an open book (with connected binding) for (Y, ξ). Ozsváth and Szabó showed:

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Under the map induced by inclusion

$$\begin{aligned} \mathbb{F}_{[x_0]} = H_*(\mathcal{F}_0) &\to \widehat{HF}(-Y) \\ [x_0] &\mapsto [x_0] =: c(Y, \xi) \end{aligned}$$

the image of $[x_0]$ is an invariant of ξ.

A refinement of the contact invariant

It follows from the discussion on the previous slide that if $c(Y,\xi) = 0$, then $[x_0]$ vanishes in the homology of some filtration level. We define

$$b(\mathfrak{ob}) = \begin{cases} \infty, & c(Y,\xi) \neq 0 \\ \min\{b \mid [x_0] = 0 \in H_*(\mathcal{F}_b)\}, & c(Y,\xi) = 0. \end{cases}$$

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Definition

Let (Y,ξ) be a contact manifold

$$b(Y,\xi) := \min\{b(\mathfrak{ob}) \mid \mathfrak{ob} \text{ an open book supporting } (Y,\xi)\}$$

Some properties

Theorem (with Baldwin)

If $\mathfrak{ob}$ is an open book decomposition with non-right veering monodromy, then $b(\mathfrak{ob}) = 1$.

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Theorem (with Baldwin)

If $\mathfrak{ob}$ is an open book decomposition with fractional Dehn twist coefficient along each binding component greater than 2, then $b(\mathfrak{ob}) > 1$.

Some corollaries

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This follows from the following structural result for knot Floer homology.

Theorem (with Baldwin)

Let $K \subset Y$ *be a non-trivial fibered knot. Then the knot Floer homology group* $\widehat{HFK}(Y, K, g(K) - 1)$ *in the next-to-top Alexander grading is non-zero.*

Some corollaries

Here's another topological result that follows almost trivially from the structure theorem for b stated on the previous slide.

Theorem (Krcatovich, with Baldwin)

If a knot $K \subset S^3$ admits an L-space surgery, then K is prime.

Extensions to transverse invariants

In joint work with Baldwin and Vértesi, we gave an alternative characterization of the transverse invariants in Heegaard Floer theory in terms of a filtration similar to the one used to define c. Thus, one can define an analogue of b which refines the usual transverse invariant.

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In a similar spirit, Diana Hubbard and Adam Salts used similar ideas to obtain a refinement of transverse invariant ψ defined by Plamenevskaya.

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This fails. Investigating why, one is naturally lead down a path toward the Heegaard Floer analogue of Algebraic Torsion.

Questions

In order to use b to effectively distinguish contact structures, it is necessary to understand how it behaves under (positive) Giroux stabilization.

Question: If $\mathfrak{ob}$ and $\mathfrak{ob}'$ are two open book decompositions for a contact manifold (Y, ξ), is $b(\mathfrak{ob}) = b(\mathfrak{ob}')$?

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Question: Under the spectral sequence from Khovanov homology to the Heegaard Floer homology of the double-branched cover is there a relationship between Hubbard and Saltz's refinement of Plamenevskaya's transverse invariant and our b?